

CHECKED

**SOLUTION
OF EQUATIONS**

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PREFACE

The solution of any problem in mathematics is accomplished by an application of one or more fundamental rules which govern the operations involved. These rules may be employed according to a definite plan, in which case the procedure is called a method of solution.

Many problems encountered in industry require a knowledge of not one, but several of the various mathematical subjects. For this reason, this text has been written to include within a single volume all of the fundamental principles of algebra, geometry, trigonometry, logarithms, and analytical geometry of straight lines.

These enumerated sections are not to be thought of as distinct non-related subjects as they are very interdependent. This fact is apparent by noting the frequent use of cross-references between the sections, without which considerable repetition would be necessary.

In practical work the labor of arithmetical computations is greatly reduced by the use of a slide rule. An appendix has been included for explaining the use of this almost indispensable tool, and it is suggested that computations be made with this aid wherever possible.

Tables of the natural trigonometric functions and the logarithms of numbers are included so as to be immediately available for reference.

In the detail preparation of this text special acknowledgment is due to William Coleal, Walter F. McGinty, Ernest J. Gentle and Morrison Perrigo for their diligent and continued efforts towards the completion of this text.

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Section I
ALGEBRA

INTRODUCTION

Algebra is simply the science of calculation by symbols and abbreviations. This branch of mathematics differs from arithmetic particularly in three ways, namely:

In algebra negative as well as positive numbers are used.

In algebra letters of the alphabet are used to represent unknown quantities and in some instances known values as well. Each letter represents a complete number regardless of the number of digits. Two or more letters appearing consecutively indicate the product of those numbers. Thus, ab means (a) multiplied by (b) .

In algebra equations are used to express the relationship between two or more quantities.

Like any form of mathematics, algebra consists of four fundamental operations—addition, subtraction, multiplication, and division. These processes are listed in the order in which they are usually found to be most easily performed. This fact should be kept in mind where a choice of solutions is possible.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used to indicate the operation to be performed:

<i>Symbol</i>	<i>Read</i>	<i>Operation</i>
+	Plus	Addition
+	Positive	
Σ	Summation	Addition
-	Minus	Subtraction
-	Negative	
\pm	Plus or minus	
\times	Times or multiplied by	Multiplication
() or [] or { }	The quantity	Multiplication
.	Multiplied by	Multiplication

In choosing one of the three symbols to indicate multiplication, the use of parenthesis in most cases is considered best because the letter x is often erroneously interpreted as an unknown term, and the period, as a decimal point.

\div or $/$	Divided by	Division
$\frac{6}{2}$ or $6/2$	6 divided by 2	Division
=	Equals	
\sim or \approx	Approximately equals	
\neq	Does not equal	
>	Greater than	

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$<$	Less than
\propto	Similar to, proportional to, varies as
∞	Infinity
\perp	Perpendicular to
\parallel	Parallel to
\dots	And so on
\therefore	Therefore
\because	Since
$\odot \odot$	Circle, circles
$\angle \angle$	Angle, angles
Rt.	Right angle
$\triangle \triangle$	Triangle, triangles
Δ	Delta
$^\circ$	Degrees
$'$	Minutes or feet
$''$	Seconds or inches
	Difference, or increment

In the following symbols and abbreviation the letter (n) is used to represent any unknown quantity or any numerical value

n_x	n sub x	No operation indicated
n_3	n sub 3	Used as shown, (x) and (3) are subscripts to distinguish one (n) from another, each being a different quantity.
n'	n prime	No operation indicated.
n''	n double prime	The prime marks are used to distinguish one (n) from another, each being a different quantity
$5n$	5 times n or $5n$	Multiplication Used as shown, 5 is termed the coefficient of (n).
$1/n$	Reciprocal of n	1 divided by (n).
$\sqrt[n]{}$	Radical sign	Extraction of root.
$\sqrt[n]{n}$	The (t) root of n	Extraction of (t) root of (n). Used as shown, (t) is termed the index of the radical.
\sqrt{n} or $n^{1/2}$	Square root of n	Extraction of square or 2nd root of n . The index 2 is customarily omitted.
$\sqrt[3]{n}$ or $n^{1/3}$	Cube root of n	Extraction of cube or 3rd root.

$\sqrt[4]{n}$ or $n^{1/4}$
 n^2

Fourth root of n
 n squared

n^3

n cubed

$\sqrt[3]{n^2}$ or $n^{2/3}$

Cube root of n^2 or square
of $\sqrt[3]{n}$ or n to the $2/3$
power

Extraction of 4th root.

Raising (n) to 2nd power
or (n) (n). Any number
or letter occupying the po-
sition of 2 as shown is
termed the exponent of (n).

Raising n to 3rd power, or
(n) (n) (n).

Raising n to a fractional
power, in this case, the $2/3$
power.

Problems involving complex exponents are most easily solved by avoiding the use of the radical sign. Writing all terms in exponential form expedites the solution.

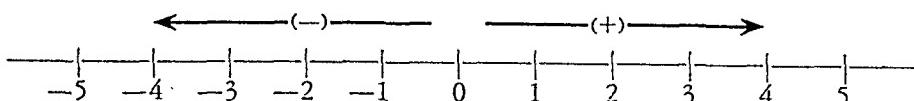
GREEK ALPHABET

Λ α Alpha	I ι Iota	P ρ Rho
B β Beta	K κ Kappa	Σ σ ς Sigma
Γ γ Gamma	Λ λ Lambda	T τ Tau
Δ δ Delta	M μ Mu	Υ υ Upsilon
E ϵ Epsilon	N ν Nu	Φ ϕ Phi
Z ξ Zeta	Ξ ξ Xi	X χ Chi
H η Eta	O \circ Omicron	Ψ ψ Psi
Θ ϑ Theta	Π π Pi	Ω ω Omega

NUMBERS

Kinds of Numbers

It has already been stated that in algebra negative as well as positive numbers are used, and, like any form of mathematics, algebra involves four fundamental operations—addition, subtraction, multiplication, and division. The rules which govern these operations when such operations are performed with positive numbers are well understood by everyone, but there is a widespread misunderstanding in regard to the use of negative numbers.



To be able to define negative and positive numbers, and to state the rules for operations with such terms it is necessary to introduce the term *absolute value* of a number. Absolute value means the numerical value of a number without regard to its algebraic sign (+) or (-). For example, +5, which is commonly written simply as 5, and -5, have exactly the same absolute values. Positive numbers are therefore absolute

values recorded positively from zero, the larger the value, the further the number is above zero. *Negative* numbers then are absolute values recorded negatively from zero; the larger the value, the further the number is below zero. When the + and -- sign of a quantity is taken into account, the term *algebraic value* is employed.

Numbers are also classified as odd or even. This term applies to the absolute value of the number only. Any number is said to be *even* if the number is divisible by 2. Consequently all other numbers are odd.

The number of figures making up a number may also be referred to as the number of *digits*, each figure of the sequence being a digit regardless of the position of the decimal point.

Such numbers as 2, 3, 12, etc. employed in the examples below are termed integers. *Integers* are defined as absolute numerical values that are complete in themselves, being whole numbers without any additional fractional parts. Obviously every whole number is an integer, or an integral number.

Two other types of numbers frequently encountered are given the descriptive titles, rational and irrational. A *rational number* is a number that *can* be represented exactly by some combination of numbers, either by an integer, a fraction (defined below) or a combination of an integer and a fraction. Thus, 30, $\frac{1}{8}$, $1\frac{1}{4}$, 1250, and 5.50 are all rational numbers.

An *irrational number* is a number that can not be represented as the quotient of two integers. Such numbers can not be expressed exactly by any combination of numbers, either by an integer, a fraction, or a combination of an integer and a fraction. The value of the mathematical symbol π , which is equal to the circumference of a circle divided by its diameter, is an irrational number, for no matter how far the computation is carried out, its value cannot be determined exactly. For practical purposes π is assumed to be 3.1416. Examples of other irrational numbers are $\sqrt{2}$, $\sqrt{3}$, etc.

Fundamental Operations With Numbers

The following rules apply to the use of positive and negative numbers. For convenience, small whole numbers are used for the purpose of explanation, but the rules apply to all quantities regardless of magnitude or whether they are numerical or algebraic terms.

Addition: The *sum* of two or more numbers is the result obtained when such numbers are added together. Numbers that are added together are called *terms*. The sum of several terms does not depend on the order in which they are added.

- (1). To add terms of like sign together, find the sum of their absolute values and prefix the proper sign (+) or (-), whichever is common to all terms.

$$\begin{array}{r}
 & & 2 \\
 2 + 3 + 4 = + 9 & & 3 \\
 & & 4 \\
 & - 2 & \\
 & - 3 & \\
 & - 4 & \\
 \hline
 & - 9 &
 \end{array}
 \quad
 \begin{array}{r}
 & & 2 \\
 & & 3 \\
 & & 4 \\
 & - 2 & \\
 & - 3 & \\
 & - 4 & \\
 \hline
 & - 9 &
 \end{array}
 \quad
 - 2 - 3 - 4 = - 9$$

- (2). To add two terms having unlike signs, subtract the term having the smaller absolute value from the term having the larger absolute value, and prefix the sign of the larger. The operation of subtraction is performed according to Rule (3).

$$\begin{array}{r} -5 \\ -5 + 3 = -2 \\ \hline +3 \\ -2 \end{array}$$

The successive application of the above two rules provides a means to add together a series of terms regardless of sign or number.

Subtraction: The *difference* between two numbers is the result obtained when one number is subtracted from the other. Subtraction is the reverse process of addition, and the numbers are likewise called *terms*. The term to be subtracted is called the *subtrahend*, and the term from which the subtrahend is subtracted is called the *minuend*.

- (3). To subtract one term from another, change the sign of the term to be subtracted, and add one term to the other according to Rules (1) and (2) above. This rule is seldom applied to positive numbers as subtracting such terms is understood by everyone. But when one or both of the terms are negative, the rule is indispensable.

$$\begin{array}{r} 5 \\ 2 \\ \hline 3 \end{array} \quad \text{or,} \quad 5 - 2 = 3$$

$$\begin{array}{r} -7 \\ -2 \\ \hline -5 \end{array} \quad \text{or,} \quad -7 - (-2) = -7 - 1(-2) = -7 + 2 = -5$$

Rule (5)

When solving the above example as written in equational form, the term 1 has been inserted in front of the parenthesis signs inclosing the -2 . This may clarify the fact that $-(-2)$ is actually equal to $+2$ since $(-1)(-2) = +2$. Rule (5). That 1 may be so inserted (actually or mentally) is apparent if it is considered that (-2) is to be taken 1 times. Also see example, Rule (26), and statement, Page 32.

$$(8a - 5b + 2c) - (4a - b + 4c) =$$

$$\begin{array}{r} 8a - 5b + 2c \\ 4a - b + 4c \\ \hline - + - \\ 4a - 4b - 2c \end{array}$$

The changed signs indicated in the above examples are for purposes of demonstration only. In practice this operation is performed mentally.

Multiplication: The *product* of two or more numbers is the result obtained when these numbers are multiplied together. Numbers that are multiplied together are called *factors*. The product of two or more factors does not depend on the order in

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which they are multiplied together. The product of zero and any number of factors is always zero. The factor which is to be multiplied is called the *multiplicand*, and the factor by which it is multiplied, the *multiplier*.

- (4) The product of *any* number of positive numbers is positive in sign. The absolute value of the product is the product of the several factors.

$$(2)(2) = +4$$

$$(2)(2)(2) = +8$$

$$(2)(n) = +2n \quad \text{Where } n \text{ is any number of positive (+) numbers.}$$

- (5) The product of any *even* number of negative (-) numbers, by themselves or together with positive numbers, is *positive* in sign.

$$(-2)(-2) = +4 \quad (\text{by themselves})$$

$$(+2)(-2)(-2) = +8 \quad (\text{with positive numbers})$$

- (6) The product of any *odd* number of negative (-) numbers, by themselves or together with positive numbers, is *negative* in sign.

$$(-2)(-2)(-2) = -8 \quad (\text{by themselves})$$

$$(-2)(+2) = -4 \quad (\text{with positive numbers})$$

Division: The *quotient* of two numbers is the result obtained when one number is divided by the other. The number being *divided* is termed the *dividend*. The number being *divided by* is termed the *divisor*. Division is the reverse process of multiplication, and the results obtained by division may be checked for accuracy, both as to sign and absolute value, by multiplying the quotient by the divisor according to Rules (4), (5), or (6) to see if it produces the original number.

- (7) The quotient obtained when one number is divided by another number of *like sign* is *positive*.

$$\frac{6}{3} = 2$$

$$\frac{-6}{-3} = +2$$

- (8) The quotient obtained when one number is divided by another number of *unlike sign* is *negative*

$$\frac{-6}{3} = -2$$

$$\frac{6}{-3} = -2$$

Rules (1) to (8) inclusive apply equally well to numerical and algebraic expressions; the numerical examples being used to clarify the rules. Furthermore, the expressions may be either integers or fractions.

Common Fractions

A fraction may be defined as the indicated quotient of two expressions such as x/y , $4/2$, $-3/5$, $\frac{a}{b-c}$. Thus, x/y means x is to be divided by y . When written in fractional form, either as x/y or $\frac{x}{y}$, the dividend (x) is termed the *numerator*, and the divisor (y) is termed the *denominator*. More simply stated, the expression above the dividing line is the numerator, and that below is the denominator. The numerator and denominator are called the terms of the fraction.

A fraction $\frac{A}{B}$ is called a *rational* fraction when both A and B are rational, a *simple* fraction when A and B are integral, and a *complex* fraction if A or B is fractional. A simple fraction, the numerator of which is of lower degree than the denominator, is called a *proper* fraction. If the degree of the numerator is equal to or greater than that of the denominator, the fraction is an *improper* fraction. An improper fraction can be reduced to an integral expression and a simple fraction.

An indicated division is often called a fraction even though the division can be performed exactly, that is without any remainder, such as $6/3 = 2$. Any integer can be made a fraction by assuming 1 as a denominator. Thus, $5 = 5/1$. The result obtained when the answer is not integral may be expressed as a simple fraction, or as an integer together with a simple fraction. Such numbers, consisting of a whole number and a fraction, are called mixed numbers. An example is:

$$x = \frac{25}{4} \text{ or } 6\frac{1}{4}$$

Fractions composed of letters are usually spoken of as algebraic fractions and those composed of definite numerical values are called numerical fractions. However, there is no difference in the manner in which they are treated in order to solve the problems in which they occur. In fact, it is a recommended practice, when in doubt about an operation in algebraic fractions, to perform a similar operation using numerical values which will allow a rapid check of the method and from this determine what the operation with the algebraic expression should be.

The operations described in the following pages are valid for both simple and complex fractions, algebraic or numerical. The examples employing algebraic fractions are paralleled by similar examples employing numerical fractions. This plan has been followed because the results of the operations with the numerical values are obvious. These results then serve to substantiate the rules as given.

- (9). The value of a fraction is not altered by multiplying *both* the numerator and the denominator by the same number or by the same (or equal) expression.

$$\frac{x}{y} = \frac{3x}{3y}$$

Since:

$$\frac{4}{2} = \frac{(3)(4)}{(3)(2)} = \frac{12}{6} = 2$$

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$$\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{5(1.414)}{2} = 3.535$$

(Note: $\sqrt{2} = 1.414$ approximately)

- (9a) The value of a fraction is not altered by changing the sign of both the numerator and the denominator. This operation is equivalent to multiplying both the numerator and the denominator by the quantity (-1) .

$$\frac{6}{2} = \frac{-6}{-2} = 3$$

- (9b) The value of a fraction is not altered by changing the sign before the fraction and also the sign of either the numerator or the denominator.

$$\frac{6}{2} = -\frac{-6}{2} = -\frac{6}{-2} = 3$$

- (10) The value of a fraction is not altered by dividing *both* the numerator and the denominator by the same number or by the same (or equal) expression

$$\frac{x}{y} = \frac{x/3}{y/3}$$

Since $\frac{12}{6} = \frac{12/3}{6/3} = \frac{4}{2} = 2$

$$\frac{3\sqrt{3}}{5\sqrt{4}} = \frac{3\sqrt{4}\sqrt{3}}{5\sqrt{4}\sqrt{4}} = \frac{(3)(2)\sqrt{3}}{(5)(4)} = \frac{6\sqrt{3}}{20} = \frac{3\sqrt{3}}{10} =$$

$$\frac{(3)(1732)}{10} = \frac{5196}{10} = 5196$$

(Note: $\sqrt{3} = 1.732$ approximately)

In dealing with common fractions, it is important to know that there is *no* rule which states that both the numerator and denominator of a fraction may be raised to any power, or that any root may be taken of both numerator and denominator without altering the value of the fraction. Such operations are frequently erroneously performed. The fallacy of such an operation is shown by an example on Page 30.

In order to solve certain types of expressions and equations involving fractions, it is necessary that all terms are written so as to have a *common denominator*. Such a denominator must be identical for each term, and of such magnitude as to include each denominator of the original terms an exact number of times. Thus 12 is a common denominator for $2/3$ and $3/4$.

It is apparent that a common denominator for two or more fractions can always be obtained by multiplying together each denominator of the several fractions, since such a product would obviously include each denominator of the original terms an exact number of times. A common denominator can also be obtained by multiplying together each *different* denominator of the several fractions. These two methods are often employed in adding or subtracting a series of fractions having dissimilar denominators.

nators. The results obtained by these methods are correct but are not always expressed in their simplest form and therefore require additional operations. To eliminate this deficiency, the denominator employed should be chosen so that it exactly contains each denominator of the given terms, yet is no larger than necessary to do so. Such a denominator is called the *least common denominator* of the several fractions, or abbreviated, L. C. D.

The method of determining the L. C. D. of two or more fractions employs the use of the term factors and prime factors which are now explained and defined:

The term *factor* is defined as a quantity which can be exactly divided into the given term. Thus 3 is a factor of 12, since $12/3 = 4$. Factors, for some purposes, are not always integers, but must be considered as integral numbers at this time. The quotient (4) obtained in this case can be further factored into (2) (2), but can be factored no further.

When any quantity is completely factored, the *prime* factors are obtained. Such factors are easily recognized since they are exactly divisible only by themselves or by one.

Example:

$$60 = (5)(3)(2)(2)$$

- (11). The value of the least common denominator for any number of fractions is readily found as follows:

Step 1. Find the prime factors of each of the denominators when each fraction is represented in its simplest form.

Step 2. Find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one denominator.

$$\frac{2}{3} + \frac{3}{4} - \frac{4}{9} = \frac{2}{3} + \frac{3}{2 \cdot 2} - \frac{4}{3 \cdot 3}$$

$$\text{L. C. D.} = (2)(2)(3)(3) = 36$$

$$= \frac{24}{36} + \frac{27}{36} - \frac{16}{36} \quad \text{Rule (9)}$$

The failure to represent each fraction in its simplest form will result in finding a number which is not the L. C. D. and the full advantage of the method will not be realized.

- (12). The sum of two or more fractions having a common denominator is a fraction whose numerator is the sum of the numerators of the fractions to be added, and whose denominator is the common denominator.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Since: $\frac{6}{2} + \frac{4}{2} = \frac{6+4}{2} = \frac{10}{2} = 5$

- (13). The difference between two fractions having a common denominator is a fraction whose numerator is the difference between the numerators of

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the fractions to be subtracted, and whose denominator is the common denominator.

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Since $\frac{6}{2} - \frac{4}{2} = \frac{6-4}{2} = \frac{2}{2} = 1$

The sum or the difference of any two or more fractions having dissimilar denominators is usually found by first reducing the given fractions to equivalent fractions all having a common denominator. The following rules, (14) and (15), however, can be used to advantage when only two such fractions are involved

- (14). The *sum* of any two fractions having dissimilar denominators, such as

a/b and c/d is $\frac{ad+bc}{bd}$. This is shown to be true by the application of

Rule (9) and Rule (12).

$$\frac{a}{b} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{bc}{bd} \quad \text{Rule (9)}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd} \quad \text{Rule (12)}$$

$$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12} = 1\frac{5}{12}$$

By rule: $a = 3, b = 4, c = 2, d = 3$

$$\frac{3}{4} + \frac{2}{3} = \frac{(3)(3) + (4)(2)}{(4)(3)} = \frac{9+8}{12} = \frac{17}{12} = 1\frac{5}{12}$$

- (15) The *difference* between any two fractions having dissimilar denominators,

such as a/b and c/d is $\frac{ad-bc}{bd}$.

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd} \quad \text{Rule (14)}$$

$$\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

By rule: $a = 3, b = 4, c = 2, d = 3$.

$$\frac{3}{4} - \frac{2}{3} = \frac{(3)(3) - (4)(2)}{(4)(3)} = \frac{9-8}{12} = \frac{1}{12}$$

- (16). The *product* of two or more fractions is another fraction whose numerator is the product of the separate numerators and whose denominator is the product of the denominators

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{ac}{bd}\right)$$

Since: $\left(\frac{4}{2}\right)\left(\frac{6}{3}\right) = \frac{(4)(6)}{(2)(3)} = \frac{24}{6} = 4$

Check $(2)(2) = 4$

- (16a). The product of a fraction and an integer is a fraction whose numerator is the product of the numerator of the given fraction and the integer, and whose denominator is the denominator of the given fraction.

$$\left(\frac{3}{4}\right)(5) = \frac{(5)(3)}{4} = \frac{15}{4}$$

Since: $\left(\frac{3}{4}\right)(5) = \left(\frac{3}{4}\right)\left(\frac{5}{1}\right) = \frac{15}{4}$ Rule (16)

- (16b). The product of a fraction and an integer is a fraction whose numerator is the numerator of the given fraction, and whose denominator is the quotient of the denominator of the given fraction and the integer.

$$\left(\frac{7}{8}\right)(4) = \frac{7}{8/4} = \frac{7}{2} = 3.5$$

Since: $\left(\frac{7}{8}\right)\left(\frac{4}{1}\right) = \frac{28}{8} = \frac{7}{2} = 3.5$ Rule (16)
Rule (16a)

- (17). The *quotient*, when one fraction is divided by another, is obtained by inverting the fraction which is the divisor and then multiplying according to Rule (16).

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

Since: $\frac{6}{3} \div \frac{2}{1} = \left(\frac{6}{3}\right)\left(\frac{1}{2}\right) = \frac{6}{6} = 1$

Check $2 \div 2 = 1$

- (17a). The quotient, when a fraction is divided by an integer, is a fraction whose numerator is the numerator of the given fraction divided by the integer, and whose denominator is the denominator of the given fraction.

$$\frac{8}{15} \div 4 = \frac{8/4}{15} = \frac{2}{15}$$

Since: $\frac{8}{15} \div 4 = \frac{8}{15} \div \frac{4}{1} = \left(\frac{8}{15}\right)\left(\frac{1}{4}\right) = \frac{8}{60} = \frac{2}{15}$ Rule (17)

- (17b). The quotient, when a fraction is divided by an integer, is a fraction whose numerator is the numerator of the given fraction, and whose denominator is the product of the integer and the denominator of the given fraction.

$$\frac{9}{2} \div 5 = \frac{9}{(5)(2)} = \frac{9}{10}$$

Since $\frac{9}{2} \div 5 = \frac{9}{2} \div \frac{5}{1} = \left(\frac{9}{2}\right)\left(\frac{1}{5}\right) = \frac{9}{10}$

In the proof of Rules (16) and (17) above, use is made of the fact that an integer can always be written as a fraction by simply assuming unity (1) as a denominator. This is obviously a valid operation and can be used whenever desired.

(18) The reciprocal of a fraction is merely the fraction inverted.

$$\frac{1}{a/b} = \left(\frac{1}{1}\right)\left(\frac{b}{a}\right) = \frac{b}{a} \quad \text{Rule (17)}$$

Since: $\frac{1}{2/4} = \left(\frac{1}{1}\right)\left(\frac{4}{2}\right) = \frac{4}{2} = 2$

Check $\frac{1}{1/2} = 2$

Decimal Fractions

The fundamental operations of addition, subtraction, multiplication, and division of numbers involving fractions can be greatly simplified if these fractions are written as decimals. Furthermore, only decimal fractions can be used for operations performed with a slide rule.

A decimal fraction is similar to a common fraction except that in a decimal fraction the denominator is always 10, 100, 1000, etc. In writing a decimal fraction it is convenient to omit the denominator and indicate its value by placing a point (.), called a decimal point, in the numerator so that there are as many figures to the right of this point as there are zeros in the denominator.

Thus, $\frac{5}{10}$ is written .5 or 0.5, $\frac{25}{100} = .25$ or 0.25,

$\frac{75}{1000} = .075$ or 0.075, etc

The zero, which is written to the left of the decimal point in many cases for clearness, is not necessary and may be omitted.

The number of figures, including zeros, to the right of the decimal point are called decimal places. It should be noted that when there are fewer figures in the numerator than there are zeros in the denominator, zeros are inserted to the right hand side of the decimal point, between the decimal point and the figures making up the decimal fraction, to make the required number of decimal places. For example:

$$\frac{75}{1000} = .075$$

It is apparent that the position of the decimal point is the factor controlling the numerical value of any given sequence of figures. For each place the point is moved to the right, the value of the decimal fraction is multiplied by 10; and for each place it is moved to the left, the value is divided by 10.

Thus, 2.5 becomes 25 when the decimal point is moved one place to the right, and 0.25 when the point is moved one place to the left. In the first case, 2.5 is multiplied by 10, and in the second case, it is divided by 10.

Multiplication and division by 100 and 1000, etc., is similarly performed. For every place the decimal point is moved toward the right, the value of the given number is increased ten times, and for every place the decimal point is moved to the left, the value of the given number is decreased ten times. Therefore, the relative values of the various places to the right and left of the decimal point are as follows:

A decimal fraction, or a mixed number consisting of a whole number and a decimal fraction, can always be obtained from a common fraction by dividing the numerator by the denominator. Zeros may be added to the numerator and the division continued until the desired degree of accuracy is attained.

The operation of changing a decimal fraction into a common fraction of the same value is not frequently required, but where necessary can be obtained by one of the following two rules.

- (19). To change a decimal into a common fraction, write the given sequence of figures without any decimal point as the numerator. The denominator is the integer 1 with as many zeros annexed at its right as there are decimal places in the given decimal fraction. In many cases, the resulting fraction can be simplified by dividing both the numerator and denominator by the same number.

$$.25 = \frac{25}{100} = \frac{1}{4}$$

- (20). To change a decimal into a common fraction having a specified denominator, multiply the decimal by a common fraction which has both a numerator and denominator equal to the specified denominator. The product is the equivalent common fraction.

$$.8125 \times \frac{16}{16} = \frac{13}{16}$$

Operations involving decimal fractions are performed according to the following rules:

- (21). To add decimals, place the numbers to be added in a column so that their decimal points are in line under one another. Then add as if they were whole numbers, and place the decimal point in the sum directly under the decimal points of the numbers being added.

SOLUTION OF EQUATIONS

$$\begin{array}{r}
 .375 \\
 .253 \\
 \hline
 .628
 \end{array}
 \begin{array}{r}
 15.32 \\
 2.756 \\
 \hline
 18.076
 \end{array}
 \begin{array}{r}
 50.057 \\
 0.023 \\
 \hline
 50.080
 \end{array}$$

- (22) To subtract one decimal from another, place the number to be subtracted below the number being subtracted from so that their decimal points are in line. Then subtract as if they were whole numbers, and place the decimal point in the difference directly under the decimal points of the numbers above

$$\begin{array}{r}
 .375 \\
 253 \\
 \hline
 122
 \end{array}
 \begin{array}{r}
 15.32 \\
 2.756 \\
 \hline
 12.564
 \end{array}
 \begin{array}{r}
 50.057 \\
 0.023 \\
 \hline
 50.034
 \end{array}$$

- (23) To multiply decimals, find their product as if they were whole numbers, and then point off as many decimal places in the product as the sum of the decimal places in the factors being multiplied together. Zeros may be inserted between the decimal point and the figures obtained as a product to make the required number of decimal places

$$\begin{array}{r}
 .2 \\
 3 \\
 \hline
 .06
 \end{array}
 \begin{array}{r}
 25 \\
 03 \\
 \hline
 .0075
 \end{array}
 \begin{array}{r}
 125 \\
 .5 \\
 \hline
 6.25
 \end{array}$$

- (24) To divide one decimal by another, find the quotient as if they were whole numbers. Zeros may be added to the dividend and the division continued until the desired degree of accuracy is attained. The position of the decimal point in the quotient can usually be found by inspection. The assumed position of the decimal point can be checked for validity by multiplying the quotient by the divisor to obtain the original dividend. In division of decimals, it is often convenient to change the divisor to a whole number by moving the decimal point to the right as many places as there are figures in the decimal. The decimal point in the dividend must be moved an equal number of places to the right, counting from the original position. If there are fewer figures to the right of the decimal point in the dividend than there are to the right of the decimal point in the divisor, annex enough zeros to the right of the dividend to take care of the new decimal point. Then divide as in whole numbers

$$\begin{array}{r}
 25 / \overline{75} \\
 \quad \quad \quad 30 \\
 25 / \overline{750} \\
 \quad \quad \quad 75 \\
 \hline
 \quad \quad \quad 0
 \end{array}$$

Square Root of Numbers

The frequent need for finding the square root of numbers makes it desirable to know a method of solution that does not require the use of tables, slide rule, or calculating machine.

The square roots of the numbers 1, 4, 9, 16, 25, 36, 49, 64, and 81, are 1, 2, 3, 4, 5, 6, 7, 8, and 9 respectively. Such numbers as those mentioned, which have exact square roots, are termed *perfect squares*. The square root of any number which is not a perfect square, such as 37, 11, etc., cannot be *exactly expressed*, although its approximate value can be found to any number of decimal places, and therefore, quite accurately.

The explanation and solution of several problems in square root, given below, indicates the procedure in finding the square root of any number, whether it be a perfect square or not. For imperfect squares, the root should be found to one more decimal place than is desired in the answer, so that the final figure of the root may be increased or left unaltered according to whether the additional figure obtained in the root is greater or less than 5.

Example 1:

$$\text{Solution: } \sqrt{1190.25}$$

Step 1.

First separate (by actual indications as shown, or mentally) the number into *periods* of two figures each, commencing at the decimal point and going both to the right and to the left. The extreme left period may consist of two digits, as in this example, or only one digit as in Example 2. The extreme right period should consist of two digits, as a zero can always be annexed where an additional figure is necessary to complete the period.

Step 2.

$$\begin{array}{r} 3 \\ \sqrt{11'90.25} \\ 9 \\ \hline 290 \end{array}$$

Find the largest perfect square which is not greater than the first period on the left (the largest perfect square contained in 11 is 9) and write its *square root* ($\sqrt{9} = 3$) *above* the first period, this number being the first figure of the required root. The square of this number ($3^2 = 9$) is placed *under* the first period and subtracted from it ($11 - 9 = 2$). Now bring down the second period (90) and annex, (not add), it to the remainder (2), thus obtaining (290), termed the *first remainder*.

Step 3.

$$\begin{array}{r} 3 \ 4 \\ \sqrt{11'90.25} \\ 9 \\ \hline 290 \\ 256 \\ \hline 3425 \end{array}$$

Take twice the part of the root already found ($2 \times 3 = 6$) and use it for a *trial divisor*, writing it at the left of the first remainder. Find how

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many times this trial divisor (6) is contained in the first remainder (290) without its right-hand figure (0). Obviously, 6 is contained in 29 four times. This number (4) is placed *above* the second period and becomes the second figure of the required root. This same number (4) is also placed to the right of the trial divisor, making the *true divisor* 64. Now multiply this true divisor (64) by the last figure placed in the root (4), and write the product ($64 \times 4 = 256$) *under* the first remainder from which it is subtracted, the difference being 34. As before, bring down and annex the next period (25) to this remainder, producing a *second remainder* of 3425.

Step 4

$$\begin{array}{r} 3\ 4\ 5 \\ \sqrt{11'90\ 25} \\ 9 \\ \hline 64 \\ 290 \\ 256 \\ \hline 685 \\ 3425 \\ \hline 3425 \end{array}$$

As in Step 3, take twice the part of the root so far found ($2 \times 34 = 68$) and use it for a second trial divisor, writing it at the left of the second remainder (3425). Find how many times this trial divisor (68) is contained in the second remainder (3425) without its right-hand figure (5). 68 is contained in 342 five times. This number (5) is placed above the third period and becomes the third figure of the required root. This same number (5) is also placed to the right of the trial divisor, making the true divisor 685. Now multiply this true divisor (685) by the last figure placed in the root (5), and write the product ($685 \times 5 = 3425$) under the second remainder from which it is subtracted, the difference being zero. Since there is no remainder, and there are no more periods in the given number (1190.25) to be brought down, the solution is complete. It is apparent the number 1190.25 is a perfect square inasmuch as it has a perfect square root.

Example 2

Solution $\sqrt{589.321} = \sqrt{589\ 321\ 000}$
 $\sqrt{5'89\ 32'10'00}$

Step 1.

First separate (by actual indication as shown, or mentally) the number into *periods* of two figures each, commencing at the decimal point and going both to the right and to the left. The extreme left period, as in this example, may consist of only one digit, but is nevertheless treated as a regular two digit period; the extreme right period should consist of two digits, as a zero can always be annexed where an additional figure is necessary to complete the period.

The *number* of decimal places in the root of a number which is not a perfect square is controlled by the number of periods of zeros annexed at the right of the number. The decimal point in the root will be *placed* so that there are as many digits to the left of the decimal point as there are whole number periods in the number of which the root is desired. In the example, the decimal point in the root will be located so as to provide two figures to the left of the decimal point, since the given number contains two whole number periods.

Step 2.

$$\begin{array}{r} 2 \\ \sqrt{5'89.32'10'00} \\ 4 \\ \hline 189 \end{array}$$

Find the largest perfect square which is not greater than the first period on the left (largest perfect square contained in 5 is 4) and write its *square root* ($\sqrt{4}=2$) *above* the first period, this number being the first figure of the required root. The square of this number ($2^2=4$) is placed *under* the first period and subtracted from it ($5-4=1$). Now bring down the second period (89) and annex (not add) it to the remainder (1), thus obtaining (189), termed *first remainder*.

Step 3.

$$\begin{array}{r} 2\ 4 \\ \sqrt{5'89.32'10'00} \\ 4 \\ \hline 189 \\ 176 \\ \hline 13\ 32 \end{array}$$

Take twice the part of the root already found ($2\times 2=4$) and use it for a *trial divisor*, writing it at the left of the first remainder. Find how many times this trial divisor (4) is contained in the first remainder (189) without its right-hand figure (9). Obviously, 4 is contained in 18 four times. This number (4) is placed *above* the second period and becomes the second figure of the required root. This same number (4) is also placed to the right of the trial divisor, making the *true divisor* 44. Now multiply this true divisor (44) by the last figure placed in the root (4) and write the product ($44\times 4=176$) *under* the first remainder from which it is subtracted, the difference being 13. As before, bring down and annex the next period (32) to this remainder, producing a *second remainder* of 1332.

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Step 4.

$$\begin{array}{r}
 2 \ 4 \ 3 \\
 \sqrt{5\ 89\ 32\ 10\ 00} \\
 4 \\
 \hline
 44 \quad 1\ 89 \\
 \quad \quad 1\ 76 \\
 483 \quad 13\ 32 \\
 \hline
 \quad \quad 14\ 49
 \end{array}$$

As in Step 3, take twice the part of the root so far found ($2 \times 24 = 48$) and use it for a second trial divisor, writing it at the left of the second remainder (1332). 48 is contained in 133 three times. This number (3) is placed *above* the third period and becomes the third figure of the required root. This same number (3) is also placed to the right of the second trial divisor, making the true divisor 483. Now multiply this true divisor (483) by the last figure placed in the root (3) and write the product ($483 \times 3 = 1449$) under the second remainder. It is obviously larger than the second remainder (1332), and cannot be subtracted from it with a positive result. Three is therefore not the third figure of the root, and this step must be repeated using a smaller number than 3.

(Whenever the product of the trial divisor and the last placed figure in the root exceeds the corresponding remainder, the root number chosen is too great, and a repetition of this step in the process is necessary using a smaller figure.)

Step 5.

Repeat Step 4 using 2 as the third figure of the root

$$\begin{array}{r}
 2 \ 4 \ 2 \\
 \sqrt{5\ 89\ 32\ 10\ 00} \\
 4 \\
 \hline
 44 \quad 1\ 89 \\
 \quad \quad 1\ 76 \\
 482 \quad 13\ 32 \\
 \hline
 \quad \quad 9\ 64 \\
 \quad \quad 3\ 68\ 10
 \end{array}$$

Steps 6, 7, . . .

Continue the process until all periods of the number have been brought down, (including zero periods annexed to provide the desired number of decimal places in the root).

The complete solution appears as follows

$$\begin{array}{r}
 2\ 4.\ 2\ 7\ 5 \\
 \sqrt{589.32100} \\
 4 \\
 44 \quad \overline{189} \\
 176 \\
 482 \quad \overline{1332} \\
 964 \\
 4847 \quad \overline{36810} \\
 33929 \\
 48545 \quad \overline{288100} \\
 242725
 \end{array}$$

Since there will be a remainder when the quantity 242725 is subtracted from 288100, it is not necessary to continue the solution any further as it is obvious that additional numbers to follow 5 can be found. The number 5 therefore, represents some value actually greater than 5. Consequently, if the answer is to be written to two places to the right of the decimal point, it will be expressed as 24.28.

Example 3:

$$\sqrt{39601}.$$

It sometimes appears that a step in the solution for the root is in error, as for example in the finding of the second figure of the square root of 39601.

$$\begin{array}{r}
 1 \\
 \sqrt{3'96'01} \\
 1 \\
 2 \quad \overline{296}
 \end{array}$$

The rule as previously stated, is to see how many times twice the first figure of the root is contained in the first remainder without its right-hand figure. In this case it would be 14 as $29/2 = 14 +$. This is impossible as there can never be a two-digit number placed in the root as one of the figures, and 9 is therefore the largest number which it is possible to use.

$$\begin{array}{r}
 1\ 9\ 9. \\
 \sqrt{3'96'01.} \\
 1 \\
 29 \quad \overline{296} \\
 261 \\
 389 \quad \overline{35\ 01} \\
 35\ 01
 \end{array}$$

Example 4:

$$\sqrt{0.091204}$$

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$$\begin{array}{r} .\ 3 \\ \sqrt{0\ 09\ 12'04} \\ \underline{9} \\ 12 \\ \underline{12} \end{array}$$

Whenever the trial divisor is not contained in the corresponding remainder without its right-hand figure, place a zero in the root, and also a zero at the right of the trial divisor, then bring down the next period, and continue as before

$$\begin{array}{r} .\ 3\ 0\ 2 \\ \sqrt{0\ 09\ 12'04} \\ \underline{9} \\ 602 \\ \underline{12\ 04} \\ \underline{12\ 04} \end{array}$$

The finding of the square root of a decimal fraction is sometimes more easily solved if the position of the decimal point is changed so that the number operated on is, at least in part, a whole number. The application of this method is as follows:

$$\sqrt{0\ 091204}$$

Rule. To obtain the square root of decimal fractions, first move the decimal point an *even number* of places to the right so that the number is expressed as some whole number between 1 and 100. Find the square root of this number and then move the decimal point *half* as many places to the left as it was moved to the right in the first place. This is the square root of the given decimal fraction.

$$\begin{array}{r} 3.\ 0\ 2 \\ \sqrt{009.1204} \\ \text{Answer} \\ \sqrt{0\ 091204} \end{array}$$

The decimal point could have been moved *any even number* of places to the right to form a whole number, and not necessarily that even number of places to produce a number between 1 and 100. After finding the numerical value of the root of the number so formed, the position of the decimal point is moved half as many places to the left as it was moved to the right in the first place. This will be the square root of the given number. The rule as first stated is recommended in preference to this latter method since slightly less labor is involved.

Example 5:

The square root of a fraction in which either the numerator or the denominator, or both, are perfect squares are solved most easily as shown below:

$$\sqrt{\frac{100}{49}} = \frac{\sqrt{100}}{\sqrt{49}} = \frac{10}{7} = 1.43 \text{ (approx.)}$$

This is somewhat less laborious in most cases than if the decimal equivalent of the fraction is found first and the square root then obtained of the resulting decimal.

If only the numerator of the fraction is a perfect square, then the logical procedure is not so evident as in the previous example. However, it is recommended that the square root of the numerator and the denominator be separately found, the square root of the perfect square being found mentally, and the square root of the other term being found by slide rule, logarithms, or by arithmetical calculation depending upon the degree of accuracy required. The value of the resulting fraction is then found by dividing the numerator by the denominator.

After finding the square root (or any *even* root such as $\sqrt[4]{}$, $\sqrt[6]{}$, etc.) the prefix \pm may be assigned to the absolute value as found since the root may be either $(+)$ or $(-)$. This is true because the product of any number of positive numbers is positive in sign, and the product of any *even* number of negative numbers, by themselves or together with positive numbers, is positive in sign.

A check on the accuracy of the absolute value of the root found from any calculation may be accomplished by squaring the computed value to see if it produces the original number. In the special case where a number ending in five is to be squared, a useful method is available which simplifies the necessary arithmetical work. The steps in the application of this method are:

Step 1. Write the given sequence of digits omitting the 5. Call the resulting number ' n '.

Step 2. Determine the product of $n(n + 1)$.

Step 3. Annex 25 to the right of the product obtained in Step 2. The number thus obtained is the square of the given number.

$$\begin{array}{r} 245 \\ \cdot 245 \\ \hline (24) (24 + 1) = (24) (25) = 600 \\ (245)^2 = 60025 \end{array}$$

EXPONENTS

The exponent of a quantity indicates the number of times the quantity is to be multiplied by itself, as:

$$a^3 = (a) (a) (a)$$

If the exponent has an absolute numerical value greater than one, it is known as a power, and if less than one, the exponent is called a root. However, in many cases the word power is extended to include fractional exponents. (This usage of the term root is not to be confused with the root of an equation defined elsewhere).

Exponents may be integers, fractions, or a combination of the two, or they also may be algebraic expressions including the logarithms of quantities. Regardless of nature, they are placed to the right and above the term which they affect. Where possible, their size should be less than the size of the term itself. The radical sign should also be thought of as an exponent as it has the same effect as though the $\frac{1}{2}$ power were indicated.

Algebraic operations involving terms to which exponents are affixed must be per-

SOLUTION OF EQUATIONS

formed in accordance with a definite set of rules if valid results are to be obtained. Because of their great importance in algebra and in logarithms (Section IV) the application of these rules should be thoroughly understood.

- (25) The range of influence of the exponent or radical sign extends only over the term to which it is adjacent.

$$2ax^2 = (2a)(x^2)$$

$$\sqrt{4a} = (\sqrt{4})(a) = \pm 2a (\pm) \text{ Rule (4) (5)}$$

- (26). An expression within parenthesis, brackets, braces, a radical sign, or underscored or underscored with a horizontal line is to be treated as a single quantity. This rule is frequently applied when dealing with exponents, but is equally important in all algebraic operations.

$$(ab)^2 = a^2b^2$$

$$(a+b)^2 = (a+b)(a+b)$$

$$\sqrt{4x^2} = \sqrt{4}\sqrt{x^2} = \pm 2x (\pm) \text{ Rule (4) (5)}$$

$$-(-6+8) = 6 - 8 = -2$$

$$\sqrt{b^2 - 4ac} = \sqrt{b^2 - (4a)(-c)} = \sqrt{b^2 + 4ac} \quad \text{Rule (5)}$$

$$\left(\frac{18}{R}\right)b = \frac{18b}{R} \quad \text{Rule (16a)}$$

- (27) Positive or negative identical terms with the same exponents are added according to the rules of positive and negative numbers, Rule (1) and Rule (2).

$$2x^2 + x^2 = 3x^2$$

- (28) Positive or negative identical terms with the same exponents are subtracted according to the rule of positive and negative numbers, Rule (3).

$$2x^2 - x^2 = x^2$$

- (29). The product of two or more like quantities (which have either like or unlike exponents) is equal to this quantity with the sum of the exponents of the factors as an exponent

$$(2)(2) = 2^{1+1} = 2^2 = 4$$

$$(-2)(-2)(-2) = -2^{1+1+1} = -2^3 = -8$$

$$(2x)(x^2) = (2)(x)(x^2) = 2x^{1+2} = 2x^3$$

- (30). The product of two or more unlike quantities, each having the same exponent, is equal to the product of these quantities with the same exponent as the exponent of the factors being multiplied together.

$$(2^3)(3^3) = (6)^3 = 216$$

$$(8)(27) = 216$$

$$(a^2)(ab^3) = a^2b^3 = (ab)^3$$

- (31). The quotient of two like quantities (which have either like or unlike exponents) is equal to this quantity with the difference between the exponents of the dividend minus the divisor as an exponent.

$$\begin{aligned} 3^3 \div 3^2 &= 3^{3-2} = 3 \\ \text{Since } 27 \div 9 &= 3 \\ x^{1.5} \div \sqrt{x} &= x^{1.5} \div x^{.5} = x^{1.5 - .5} = x \end{aligned}$$

- (32). The quotient of two unlike quantities, each having the same exponent, is equal to the quotient of these quantities with the same exponent as the exponents of the quantities being divided.

$$\begin{aligned} (4)^2 \div (2)^2 &= (4/2)^2 = 4 \\ \text{Since } 16 \div 4 &= 4 \\ x^2 \div y^2 &= (x/y)^2 \end{aligned}$$

- (33). Rules (31) and (32) may be combined into a single rule which has the advantage of being more understandable and workable in many cases:

When changing any quantity from the numerator to the denominator of a simple fraction, or vice versa, the plus or minus (\pm) of the exponent of that term is reversed. The fraction can then be simplified according to the rules applying to multiplication, Rules (29) and (30).

$$\begin{aligned} \frac{x^3}{x^2} &= (x^3)(x^{-2}) = x^1 \\ \frac{b^{-2}}{2a^{-1}} &= \frac{a}{2b^2} \end{aligned}$$

The above rule can be expanded to include other than simple fractions provided that the numerators and denominators of such terms are products of terms. Such a fraction excludes any terms in addition or subtraction.

$$\begin{aligned} \frac{x^2y}{2x} &= \frac{xy}{2} \\ \frac{3x^{-2}}{a^{-1}} &= \frac{3a}{x^2} \end{aligned}$$

Rule (33) is useful in establishing the numerical value to be assigned to any term raised to the zero power. It is common knowledge that any quantity divided by itself is one.

$$\frac{x}{x} = 1$$

Also by the above rule:

$$\frac{x^1}{x^1} = (x^1)(x^{-1}) = x^0$$

Therefore since $x/x = 1$, then x^0 must also = 1, since $x/x = x^0$. (Things equal to the same thing, or equal things, must be equal to each other.)

SOLUTION OF EQUATIONS

- (34) Any quantity raised to any power is equal to the quantity with its original exponent multiplied by the power in question.

$$(2)^2 = (2^1)^2 = 2^{(1)(2)} = 2^2 = 4$$

$$(2^2)^3 = 2^{(2)(3)} = 2^6 = 64$$

$$\text{Since } (4)^3 = 64$$

- (35) Any quantity from which a given root is to be extracted is equal to the quantity with its exponent divided by the root in question.

$$\sqrt[3]{2^6} = 2^{6/3} = 2^2 = + 4$$

$$\sqrt[3]{64} = + 4$$

$$\sqrt{x} = x^{1/2}$$

$$\sqrt[n]{x} = x^{1/n}$$

WRITING OF EQUATIONS USING SYMBOLS AND ABBREVIATIONS

Using the conventional symbols and abbreviations as described, it is possible to reduce to algebraic form any problem in which there exists a mathematical relationship. Such forms may be classed as either expressions or equations. An algebraic *expression* in its simplest form is a statement that some term exists, examples of which are a , x , and 2 . In other cases an expression denotes that some operation is to be performed, such as $\sqrt{2}$, $(a - b)$, $(x - 3c)$. An *equation* is defined as a statement of equality between two expressions, that is, that two expressions are equal. $x = y$, $x - y = 25$, $a = o$. An equation can always be distinguished from an expression by the presence of the equal sign together with an accompanying term, even though this term is zero. Without this accompanying term an equation becomes an expression, thus $(x - 7 =)$ is an expression no different from $x - 7$.

Equations may be either *conditional equations* or *identities*. An equation that is true for only certain values of the unknown involved is termed a conditional equation. Identities are equations which are satisfied by all values assigned to the unknown. Identities cannot be solved because, when simplified, they become $0 = 0$. At the opposite extreme from identities stand *descriptions* which are not true equations because they cannot be satisfied by any number. When the word equation is used without further qualification, it is a conditional equation that is implied.

Equations are also classified according to their degree. The degree of an equation containing only one unknown is the same as the numerical value of the largest exponent of that unknown. The equation $x = 4$ is an equation of the first degree. Such equations are also termed linear equations since all of its plotted points will fall on a straight line. The equation $ax^2 + bx + c = 0$ is called a quadratic equation or an equation of the second degree. An equation containing x^3 is called a cubic equation or an equation of the third degree.

Whenever an unknown quantity is to be represented in either an expression or an equation it is customary to assign one of the last letters of the alphabet, as x , y , or z to indicate this quantity. Numerical quantities of known value are sometimes represented by the first letters of the alphabet, a , b , c , . . . , but it is generally advisable to use the known values directly until doing otherwise is justified by experience. The assignment of letters to represent unknown quantities as well as the proper use of known values may be shown by an example:

4 times the square of a certain unknown number minus 9 times that number is equal to — 2

Let x represent the unknown number

$$4x^2 - 9x = -2$$

It is evident that the equation expresses by a few symbols all that was formerly conveyed by a great number of words. This form also presents less possibility for misinterpretation of facts. The greatest advantage of the algebraic representation is that it allows a rapid and precise solution of the problem in question.

Rules cannot be definitely set forth in the writing of equations as they can in the methods of solution. To write an equation, the conditions of the problem must be understood, and also, the person writing the equation must know the signs and symbols—the algebraic language. This second requisite will be acquired with the study of the subjects throughout the text.

The following suggestions may be helpful as a guide when attempting to write an algebraic equation representing a mathematical relationship.

Step 1.

Read carefully the statement of the problem as given in words.

Step 2.

Determine what is the unknown quantity or quantities, and represent these by letters of the alphabet, always using the minimum number of letters. This is accomplished by expressing as many as possible of the unknown quantities in terms of one of the unknowns. For instance, if one term B is twice as large as another term A , it is represented as $2A$.

Step 3.

If all of the unknown quantities cannot be expressed in terms of one of the unknowns (as suggested above) then as many equations must be written as there are different letters used to represent the unknowns. Each of these equations must represent a separate, independent relationship. After the necessary number of equations has been obtained they may be solved, simultaneously, and the values of the several unknowns determined.

SOLUTION OF EQUATIONS

After writing an equation, or having been given an equation, the next step is to find the values of the unknowns. This procedure is referred to as solving the equation, finding the roots of the equation, or finding the values of the variables. The term roots and variables have specific meanings in some instances, but in any case their numerical values if correctly obtained are the values of the unknowns. Consequently the terms roots, variables, and unknowns are used interchangeably.

Before proceeding with the solution of any algebraic equation or equations it is important to determine if the necessary amount of information is given so that a solution is theoretically possible. If no unknowns appear in the expressions being considered, then the solution is simply a process of simplification—no unknown values are to be found. However, if the value of one unknown is to be found there must be one equation involving that particular unknown and incorporating no other. If the

number of unknowns is greater than one, it will be necessary to have as many independent equations as there are unknowns to be found. These equations, equal in number to the unknowns, must be solved simultaneously—a simple operation once the solution of a single equation is understood.

It is also important to know that the number of values which can be found to satisfy a single equation of one unknown will be equal to the highest power of the unknown appearing in that equation. The *power* of the unknown is the value of the exponent affixed to that variable. Exponents greater than unity are termed powers; less than unity, roots. The equation $4x^2 - 9x = -2$ contains but one variable since x^2 and x are the same unknown though raised to different powers. This one equation is solvable and two answers may be expected as a result.

Unfortunately, the detailed description of many operations used in solving equations makes it appear that the labor involved is considerable, when actually, the method is surprisingly brief and simple. It is therefore suggested that the examples given be examined, step by step, along with the descriptive notes. This will assist in correlating the associated theory with its practical application. After a time, short-cuts may be discovered which will make possible more direct solutions. Also, it will be possible to choose the most desirable method whenever there are several which may be employed.

Valid Operations with Equations

The rules and operations (1) to (35) inclusive have been described as applying to expressions. However, since equations are statements of equality between expressions, these same rules are also used in the solution of equations. In addition, certain operations are usually required in which the entire equation and not just one of its expressions are involved. Since equations consist of two sides separated by the equal sign, it is possible to define these additional rules and operations as applying to *both sides* of the equation, even though one of these sides is zero. This emphasis on *both sides* helps to eliminate the possibility of applying a rule or operation to only one side of an equation, and leaving the other side unaffected.

In the solution of equations, the following operations may be performed without destroying the equality of the relationship.

- (36) The same or equal quantities may be added to both sides of an equation

$$\begin{array}{rcl} 3 &=& 3 \\ 2 + 3 &=& 3 + 2 \\ 5 &=& 5 \end{array}$$

- (37). The same or equal quantities may be subtracted from both sides of an equation

$$\begin{array}{rcl} 3 &=& 3 \\ 3 - 2 &=& 3 - 2 \\ 1 &=& 1 \end{array}$$

- (38). In an equation involving addition or subtraction, any term may be transposed from one side of the equal sign to the other provided that its plus or minus sign is reversed

$$\begin{array}{rcl} 4 + 2 &=& 6 \\ 4 &=& 6 - 2 \end{array}$$

An analysis of Rule (38) will show that it, in reality, embraces the same principles as are set forth in Rules (36) and (37).

Transposing and changing sign is often erroneously applied to equations in which the various terms appear as products or quotients. The fallacy of such an operation is apparent if a simple problem is involved:

$$\begin{array}{rcl} 6 = \frac{12}{2} & & (2)(3) = 6 \\ -\frac{6}{2} = 12 & & 3 = 6 - 2 \\ & & 3 \neq -4 \\ -3 \neq 12 & & \end{array}$$

Both solutions are obviously *incorrect*. The given equations were not in addition and subtraction, but involved division and multiplication respectively, and hence the rule does not apply.

- (39). Both sides of an equation may be multiplied by the same or by an equal quantity.

$$\begin{array}{l} 3 = 3 \\ (2)(3) = (2)(3) \\ 6 = 6 \\ 5 + 2 = 7 \\ 2(5 + 2) = (2)(7) \end{array}$$

Expanding or performing the indicated multiplication of the above example may be accomplished by either one of two methods:

$$\begin{array}{lll} (2)(5) + (2)(2) = 14 & \text{or} & 2(7) = 14 \\ 10 + 4 = 14 & & 14 = 14 \\ 14 = 14 & & \end{array}$$

- (40). Both sides of an equation may be divided by the same or by an equal quantity. (Zero can never be used as a divisor as demonstrated on Page 50.)

$$\begin{array}{l} 8 = 8 \\ 8/4 = 8/4 \\ 2 = 2 \\ 4 + 6 = 10 \\ \frac{4+6}{2} = \frac{10}{2} \end{array}$$

Performing the indicated division for examples of this type may be accomplished by several methods:

$$\begin{array}{lll} \frac{4}{2} + \frac{6}{2} = 5 & \text{or} & \frac{10}{2} = 5 \\ 2 + 3 = 5 & & 5 = 5 \\ 5 = 5 & & 2 + 3 = 5 \\ & & 5 = 5 \end{array}$$

SOLUTION OF EQUATIONS

This example as solved by the method on the right demonstrates the fact that the quotient obtained by dividing one quantity by another is the same as the product of the reciprocal of the divisor and the quantity to be divided.

- (41). The reciprocal may be taken of both sides of an equation

$$4/2 = 6/3 \quad \text{or} \quad 2 = 2$$

$$\frac{1}{4/2} = \frac{1}{6/3} \quad \text{or} \quad \frac{1}{2} = \frac{1}{2}$$

The reciprocal of a fraction is merely the fraction inverted, thus if the equation is given,

$$\frac{1}{x} = \frac{5}{3}$$

Then $\frac{x}{1} = \frac{3}{5}$ or $x = \frac{3}{5}$ Rule (18)

The reciprocal of both sides of an equation in which one side consists of two or more fractions is found as follows

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$$

$$R = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

$$R = \frac{1}{\frac{b+a}{ab}}$$

$$R = \frac{ab}{b+a} = \frac{ab}{a+b} \quad \text{Rule (17) or (18)}$$

- (42). Both sides of an equation may be raised by any identical power. In some cases this operation may cause additional roots to be formed which will not check when substituted into the given equation. This is explained in the paragraphs entitled Radical Equations.

$$\begin{array}{ll} 4 = 4 & x^{3/2} = 8 \\ 4^2 = 4^2 & (x^{3/2})^2 = 8^2 \\ 16 = 16 & (x^{3/2})(\sqrt[3]{8^2}) = x^{6/2} = 64 \\ & x^3 = 64 \\ & x = +4 \end{array}$$

- (43). Any identical root may be extracted from both sides of an equation. The odd roots of any number are positive, and the even roots are positive or negative according to Rules (4) and (5).

$$\begin{array}{l} 25 = 25 \\ \sqrt{25} = \sqrt{25} \\ \pm 5 = \pm 5 \end{array}$$

$$\begin{array}{l} 27 = 27 \\ \sqrt[3]{27} = \sqrt[3]{27} \\ + 3 = + 3 \end{array}$$

(44). The logarithm may be taken of both sides of an equation.

$$\begin{array}{l} x = y \\ \log x = \log y \end{array}$$

The application of this rule will be demonstrated in Section IV—
Logarithms.

In calculus, the derivative and the integral of both sides of an equation may be taken.

Practical Check on Algebraic Operations

Whenever the validity of an operation in the solution of an algebraic equation is questionable, it is a good practice to set up a simple example similar to the problem involved, but using small integers in place of more complicated terms. By using values such that the correct answer is apparent by inspection, the questionable operation performed in the simple example will set forth the procedure necessarily applied in the original more complicated equation.

The several examples shown below are but a few of the many applications of this method. In choosing integers for substitution, the use of the values 1 and 2 is to be avoided unless it is definitely known that these values will not satisfy an operation which for any other integers would be an erroneous process.

Is $\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2}$
Let $a = 4$ $b = 3$

$$\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = \pm 5$$

Assume $\sqrt{4^2 + 3^2} = \sqrt{16} + \sqrt{9} = (\pm 4) + (\pm 3) = \pm 7; \pm 1$

The questionable operation is obviously incorrect.

Is $\frac{m}{n} = \frac{x}{y}$ equivalent to $(m)(y) = (n)(x)$

$$(ny) \left(\frac{m}{n} \right) = (ny) \left(\frac{x}{y} \right)$$

$$my = nx \text{ Multiplying by } (ny)$$

Rule (39)

Also $\frac{6}{3} = \frac{4}{2}$

$$(6)(2) = (3)(4)$$

$$12 = 12$$

The questionable operation is correct. This operation is known by some as: In a proportion the product of the mean quantities is equal to the product of the extremes. This relationship may be referred to as the diagonal rule, or simply that the cross-products of a proportion are equal. (See page 35.)

Is $\frac{\sqrt{2}}{2} = \frac{(\sqrt{2})^2}{2^2}$

$$\frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707$$

$$\frac{(\sqrt{2})^2}{2^2} = \frac{2}{4} = 5$$

The questionable operation is obviously incorrect. There is *no* rule which states that both numerator and denominator of a fraction may be squared (See Rule (42) for comparable operation involving equations).

Is $\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$$\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{(2)(1.414)}{2} = 1.414 \quad \text{Rule (9)}$$

The operation as shown above is correct, being an application of Rule (9). This simplification is frequently used to change the denominator to an integer and thus a more convenient divisor.

Is $-\frac{6x - 4}{2} = -3x - 2 \text{ or } -3x + 2$

$$-\frac{6x - 4}{2} = -3x + 2 \quad \text{Rule (26)}$$

Because, Let $x = 2$

$$-\frac{6x - 4}{2} = -\frac{12 - 4}{2} = -\frac{8}{2} = -4$$

$$-3x - 2 = -6 - 2 = -8 \quad (\text{incorrect})$$

$$-3x + 2 = -6 + 2 = -4 \quad (\text{correct})$$

Is $(b^2)(b^3) = b^{(2+3)} \text{ or } b^{2+3}$

$$(b^2)(b^3) = (b^5)$$

Because,

$$\text{Let } b = 2$$

$$(2^2)(2^3) = (4)(8) = 32$$

$$(2^2) = 2^{(2+3)} = 2^6 = 64 \quad (\text{incorrect})$$

$$(2^2)(2^3) = 2^{2+3} = 2^5 = 32 \quad (\text{correct})$$

METHODS OF SOLUTION—SINGLE EQUATION

The solution of any algebraic equation is accomplished by an application of one or more fundamental rules which govern the operations involved. These rules may be employed according to a definite plan, in which case the procedure is called a method of solution.

In explaining the various methods of solution the term *unknown* and the expression *degree of an equation* are frequently used. These have been previously defined.

The word *coefficient*, not previously employed by name, will be used in the following pages. This term refers to the prefix of the unknown term written to indicate how many times the unknown is to be taken as a factor; thus $5x$ means 5 times (x) or the quantity $5x$. As in this example, the coefficient is usually a numerical value although not necessarily an integer. It is sometimes represented by one of the first letters of the alphabet as a , b , c .

Other numerical values in an equation unattached to unknowns are called *constant terms*. For example, $5x^2 + 10x + 15 = 20$ contains two constant terms, 15 and 20, which may be combined into a single value of ± 5 depending on which side of the equation the terms are collected according to Rule (38). The name constant is appropriate to such terms since numerical values are of constant or unvarying magnitude. In this same equation the coefficients 5 and 10 are also constants from this viewpoint, but should be considered as coefficients inasmuch as they are attached to unknowns.

Equations varying from the simple to the complex are sometimes referred to as *polynomials*, meaning many numbers. However, this is not the principal use of this term, since a polynomial is any algebraic expression of two or more terms, as $(x + 2)$, $(a + b + c)$. Another word, *function*, is also used to mean an equation. This latter term will be used in preference to the word polynomial when a substitute term for equation is desired. This will allow the meaning of polynomial to be restricted to the use of expressions only.

The following methods of solution are applications of the rules and operations described in the preceding pages. In many cases the answer or answers are obtained by the performance of a single operation, but to the other extreme, a great deal of computation may be required. Furthermore, for certain types of problems, the results may be obtained by following two or more distinct procedures. The applications and relative merits of each of these solutions are not all at first evident. The *best* method of solution will also be a matter of the user's personal likes and dislikes. Where a choice in the method of solution exists it is only by experience that the most convenient and practical form or forms will become apparent.

Simplification

Any equation of the first degree involving only *one* unknown appearing any number of times can be solved by means of simplification:

$$\begin{aligned} 2x - 2 &= 4x - 4 \\ +4 - 2 &= 4x - 2x \\ 2 &= 2x \\ x &= 1 \end{aligned} \qquad \text{Rule (38)}$$

For no apparent reason it is customary to transpose and collect all terms of the variables on the left side of the equal sign and to do likewise with the constant terms on the right side. There is no fallacy in this as it is possible to use either side for these terms. However, it seems logical to transpose all of the variable terms to that side where, when collected, their sum will be a positive (+) quantity. In the example above this happens to be the right-hand side, and by so doing one step in the solution may be omitted. In the last step, it makes no difference if the equality is written:

$$\begin{aligned} 1 &= x \\ \text{or} \quad x &= 1 \end{aligned}$$

In many cases a group of terms are included within parentheses to show that the group is to be treated as a single quantity. To solve such an equation it is necessary to first remove such parentheses and consolidate the terms as much as possible.

$$\begin{aligned}x + 3(2 - a) &= 2a \\x + 6 - 3a &= 2a \\x = 2a + 3a - 6 &= 5a - 6\end{aligned}$$

Although an extremely simple example, the above equation demonstrates the fact that a parenthesis sign preceded by a positive factor has no effect upon the sign (\pm) of the enclosed terms when the parentheses are removed. The same holds true when no term precedes the parenthesis other than the + sign. In this case the quantity is to be taken + 1 times, although the number 1 is not shown, merely inferred.

The presence of either a (—) sign or a negative factor preceding an expression enclosed within parentheses serves to reverse the sign of each term of the expression when the parentheses are removed. The procedure involved in the removal of parentheses is based directly on the rules governing the products of positive and negative numbers.

The accompanying example shows the application of these rules and also the occurrence of a parentheses within a parentheses. In such cases it is recommended procedure to commence with the innermost expression first and remove one pair of parentheses at a time.

$$\begin{aligned}2x - (-2a - (a - b) + b) + 4a &= 7a \\2x - (-2a - a + b + b) + 4a &= 7a \\2x + 2a + a - 2b + 4a &= 7a \\2x + 7a - 2b &= 7a \\2x = 7a - 7a + 2b &= 2b \\x &= b\end{aligned}$$

The solution of an algebraic equation involving fractions is generally simplified if the equation can be rearranged so that all fractions are removed and all numerical values appearing in the resulting equation are integers. One method by which this can be accomplished is to first change all terms of the given equation to a common denominator, after which this denominator is discarded according to Rule (39): The value of an equation is not changed by multiplying both sides by the same or an equal quantity.

The operation as described above is in effect the same as multiplying the original equation by a quantity known as the least common multiple, abbreviated, L. C. M.

- (45). The value of the L. C. M. is the same as that of the L. C. D., and is found by an identical method.

Step 1.

Find the prime factors of each of the denominators when each fraction is represented in its *simpliest form*. (See second example).

Step 2.

Find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one denominator. This is the L. C. M. of the given equation.

$$\begin{aligned}x/2 + x/6 &= 2 \quad \text{or,} \quad x/2 + x/2 \cdot 3 = 2/1 \\ \text{L. C. M.} &= (2)(3) = 6 \\ 6x/2 + 6x/6 &= 12/1 \\ 3x + x &= 12 \\ 4x &= 12 \\ x &= 3\end{aligned}$$

The elimination of fractions from an equation by multiplying each term of both sides by the L. C. M. is more direct than the previously described method, consequently it should be employed in all such cases.

In determining the L. C. M. the failure to represent each fraction in its simplest form may lead to an erroneous answer as demonstrated in the following example:

$$\frac{4x - 12}{x - 3} + \frac{6}{x - 2} = 5$$

Assuming $(x - 3)(x - 2)$ to be the L.C.M.

$$\begin{aligned}(x - 2)(4x - 12) + 6(x - 3) &= 5(x - 3)(x - 2) \\ 4x^2 - 20x + 24 + 6x - 18 &= 5x^2 - 25x + 30 \\ x^2 - 11x + 24 &= 0\end{aligned}$$

The solution of this equation is accomplished by methods described in the solution of Quadratic Equations, a subject reserved for later study. However, according to elementary theory, any equation such as $x^2 - 11x + 24 = 0$ should have two answers for the value of x . For any given equation these values are usually different in absolute value and often times in algebraic sign as well. But, in many cases they may have identical values in every respect. Regardless of the nature or number of the answer obtained, the values resulting from the solution of the equation should satisfy the given equation when each of such numbers is substituted into the equation in place of the unknown quantity. If the answer obtained fulfills this requirement, it can be called a *root* of the equation. The distinction between a root and an answer is that a root is an answer known to be correct for the given equation. Many answers, even though determined by correct algebraic methods, will not be roots of the given equation.

The roots of the equation $x^2 - 11x + 24 = 0$ are found to be 3 and 8. The solution of this equation is accomplished by methods considered later, but the validity of the roots can be easily ascertained. However, 3 cannot be a root of the given equation,

$$\frac{4x - 12}{x - 3} - \frac{6}{x - 2} = 5$$

as it makes the first fraction zero and the value of the second fraction — 6 which destroys the equality of the relationship. The equation $x^2 - 11x + 24 = 0$ is, therefore, not equivalent to the given equation. An inspection of the first fraction of the given equation reveals that it is not in its simplest form since

$$\frac{4x - 12}{x - 3} \text{ is } \frac{4(x - 3)}{x - 3} \text{ or } 4.$$

Therefore, the L.C.M. is $(x - 2)$, and the simplified equation is:

SOLUTION OF EQUATIONS

$$\begin{aligned}4(x-2) + 6 &= 5(x-2) \\4x - 8 + 6 &= 5x - 10 \\10 - 8 + 6 &= 5x - 4x \\8 &= x\end{aligned}$$

By substitution 8 is found to be a root of the given equation. Note that the unknown quantity in this case is in the denominator, and that this is the only place where it occurs.

The importance of using each fraction in its simplest form when determining the L.C.M. for the given equation is apparent from the example above. From this it might be implied that the use of the L.C.M. is essential to the solution of equations involving fractions. This is not true, however, because in most cases the roots of the equation resulting from clearing the original equation of fractions are the same as the roots of the original equation, whether the L.C.M. is used or not. This will always be true when no denominator of the original equation contains the unknown. However, the elimination of fractions from an equation by multiplying each term of both sides by the L.C.M. is the most direct solution and should be employed in all cases.

When the variable appears in the denominator of one or more of the terms of the original equation, it is quite possible that the equations formed by clearing the original equations of fractions by multiplying each term by the L.C.M., will not be equivalent to the given equation, and the resulting equation may have roots that are not roots of the given equation. Such roots which are found by correct algebraic means, yet will not check in the original equation are called *extraneous roots*. It is, therefore, necessary to determine by substitution whether any roots of this simplified equation are not roots of the original equation. If they are not, this fact is at once apparent, for if such a root is substituted in the original equation it will make some denominator of the original equation equal to zero. A root of this kind cannot be a root of the original equation.

The simplification of some equations involving fractions may lead to an equation of the form

$$\frac{2}{3}x = 6$$

Assuming that it is the value of (x) that is desired, the above equation can immediately be solved by multiplying both sides of the equation by $3/2$ or the reciprocal of the coefficient of the unknown term.

$$\left(\frac{3}{2}\right)\left(\frac{2}{3}x\right) = \left(\frac{3}{2}\right)(6) \quad \text{Rule (39)}$$

$$x = 9$$

To find the reciprocal of any number simply divide one by that number. The reciprocal of a fraction is merely the fraction inverted. Thus the product of any number and its reciprocal is always +1.

The use of the above method in preference to the use of the L.C.M. is left to the user's discretion. Both methods are correct.

Equations taking the form of proportions (described in the paragraphs titled Ratio, Proportion and Variation) are so frequently encountered that it is advisable to show

by an example the most practical solutions. The position of the unknown, x , may be in any one of the four possible positions.

$$\frac{a}{b} = \frac{x}{c}$$

$$\frac{ac}{b} = x \text{ Multiplying by (c)}$$

Rule (39)

Another solution involving one additional step yet more commonly used than the above makes use of the fact that the cross-products of a proportion are equal. This relationship may be referred to as the diagonal rule. This operation in Geometry is stated: In a proportion the product of the mean quantities is equal to the product of the extremes.

$$\frac{a}{b} = \frac{x}{c}$$

$$bx = ac$$

$$x = \frac{ac}{b}$$

Multiplying by (c) and (b)
at the same time. Rule (39)

An equation containing only one unknown with that unknown occurring only once and with an exponent other than unity (either a root or a power) can be solved by changing the entire equation to the desired power of the unknown. Assuming that it is the value of x^1 that is desired:

$$x^2 = 16$$

$$x = \sqrt{16} = \pm 4 \text{ Rule (37)}$$

$$3x^3 = 64 + 2x^3$$

$$x^3 = 64$$

$$x = +4$$

$$\text{or } x^{3/2} = 8$$

$$(x^{3/2})^{2/3} = 8^{2/3} = (8^{1/3})^2$$

$$x^{6/6} = (2)^2$$

$$x = 4$$

$$x^{3/2} = 8$$

$$(x^{3/2})^{2/3} = 8^{2/3} = (8^2)^{1/3}$$

$$x^{6/6} = (64)^{1/3}$$

$$x = \sqrt[3]{64} = +4$$

The use of the fact that the product of a fraction and its reciprocal is always $+1$ is of distinct advantage in the solution of this problem. That this is true is apparent if the solution of the same problem by another method is examined.

Many equations include a number of terms arranged to indicate that more than one operation is to be performed. An unlimited number of such examples can be shown, but only two arbitrarily chosen types will demonstrate the fact that only basic principles included in Rules (1) to (45) inclusive are involved.

The equation ($L = C \frac{r}{2} SV^2$), solved for V , presents a type of solution frequently required in practical work.

$$2L = CrSV^2$$

$$\frac{2L}{CrS} = \frac{CrSV^2}{CrS} \quad \text{Rule (40)}$$

$$\frac{2L}{CrS} = V^2 \quad \text{Rule (10)}$$

The three factors C , r and S by which both sides of the equation are being divided are not ordinarily written down as a denominator on the side of the equation from which these terms are to be eliminated. This is obviously a waste of effort as the quotient in such cases will always be 1. However, in the example above, this step has been performed to emphasize that to isolate the unknown term from any factors which may accompany it, simply divide both sides of the equation by these factors.

$$\frac{2L}{CrS} = V^2 \quad \text{Rule (10)}$$

$$V = \sqrt{\frac{2L}{CrS}} \quad \text{Rule (43)}$$

The second example is of a more complex nature. Fortunately such types are infrequently encountered

$$a = \frac{b}{1 + \left(\frac{18}{R}\right)b}$$

For purposes of explanation, assume that R is the unknown quantity, and that (a) and (b) are arbitrary constants to which any one of an unlimited set of numerical values may be assigned.

The value of R is then to be determined in terms of (a) and (b) . The solution as given is not necessarily written in its briefest form, as it is intended to clearly show each operation involved.

$$a = \frac{b}{1 + \frac{18b}{R}} \quad \text{Rule (16a)}$$

$$a = \frac{b}{\frac{R + 18b}{R}} \quad \text{Rule (9), (11)}$$

$$\frac{a(R + 18b)}{R} = b \quad \text{Rule (39)}$$

$$\frac{R + 18b}{R} = \frac{b}{a} \quad \text{Rule (40)}$$

$$1 + \frac{18b}{R} = \frac{b}{a}$$

$$\frac{18b}{R} = \frac{b}{a} - 1 = \frac{b - a}{a} \quad \text{Rule (11), (38)}$$

This solution may be completed by alternate methods.

$$R(b-a) = 18ab$$

$$R = \frac{18ab}{b-a}$$

Rule (40)

$$\frac{R}{18b} = \frac{a}{b-a}$$

Rule (41)

$$R = \frac{a(18b)}{b-a}$$

Rule (39)

$$R = \frac{18ab}{b-a}$$

Trial and Error

This method, which in a few cases is the most rapid solution, is limited in its application to simple equations, or to more involved equations for which only an approximate answer is required. Trial and error methods are also used in establishing the factors of an equation as explained in the paragraphs entitled Factoring.

Trial and error consists of assigning an arbitrary number to the unknown and determining, by substitution, if the assumed number is a root of the equation. A root of an equation is a number which when substituted for the unknown quantity will make the two sides of the equation equal, that is, it satisfies the equation.

The work involved in the solution of an equation by this method is reduced if the given equation is first simplified. Furthermore, it is advisable to transpose all terms of the equation to one side of the equal sign with the other side of the equation becoming zero.

$$5x^2 + 10x + 15 = 30$$

$$x^2 + 2x + 3 = 6$$

$$x^2 + 2x - 3 = 0$$

Rule (40)

Rule (38)

The given equation is now expressed in its simplest form and with all terms transposed to one side of the equation. For the purpose of explanation, and not necessarily the most logical procedure, the value of the unknown is first assumed to be 2.

$$\begin{aligned}(2)^2 + 2(2) - 3 &= 0 \\ 4 + 4 - 3 &= 0 \\ 5 &= 0 \\ x &\neq 2\end{aligned}$$

Assume $x = 3$:

$$\begin{aligned}(3)^2 + 2(3) - 3 &= 0 \\ 9 + 6 - 3 &= 0 \\ 12 &= 0 \\ x &\neq 3\end{aligned}$$

As the error is increasing with increasing values assumed for (x), it is reasonable to assume that the value of the unknown is smaller and not greater than the first assumed value for the unknown which was 2. This reasoning is not always valid as may be seen by an examination of the curve of the second example in the paragraphs entitled Graphical Solution, page 42.

Assume $x = 1$.

$$\begin{aligned}(1)^2 + 2(1) - 3 &= 0 \\ 1 + 2 - 3 &= 0 \\ 0 &= 0 \\ x &= 1\end{aligned}$$

The given equation is of the second degree and consequently has two values to be determined. It is logical to assume that this second value is a negative number since it was shown above that the larger the value assumed for (x), the greater the magnitude of error. Therefore, the next value arbitrarily assumed for the unknown is -2 .

$$\begin{aligned}(-2)^2 + 2(-2) - 3 &= 0 \\ 4 - 4 - 3 &= 0 \\ -3 &= 0 \\ x &\neq -2\end{aligned}$$

It is apparent that $x = -2$ is not a root of the equation. An inspection of the equation shows that any negative value substituted for the unknown becomes positive for the x^2 term and remains negative in the x term. The constant -3 is unaffected by values assumed for the unknown. Since the value of the equation is negative for a value of $x = -2$, a larger negative value or $x = -3$ will be assumed next because such a number will make the equation as a whole increase in positive value.

$$\begin{aligned}(-3)^2 + 2(-3) - 3 &= 0 \\ 9 - 6 - 3 &= 0 \\ 0 &= 0 \\ x &= -3\end{aligned}$$

The two roots of the given equation,

$$5x^2 + 10x + 15 = 30$$

or its simplified equivalent,

$$x^2 + 2x - 3 = 0$$

are, therefore,

$$x = 1 \text{ and } x = -3$$

Additional examples involving the use of the trial and error method might be given, but they would only show the tediousness of this procedure because the variation of the value of the equation with the variation in the value assumed for the unknown is difficult to predict in many cases. The graphical solution, to be described in the following paragraphs, shows the variation of the value of the equation as the value of the unknown varies, and in this respect alone justifies its preference to the trial and error method. The graphical solution has an additional benefit in that it indicates approximately the value of any irrational roots which may exist, a feature which becomes exceedingly laborious using trial and error.

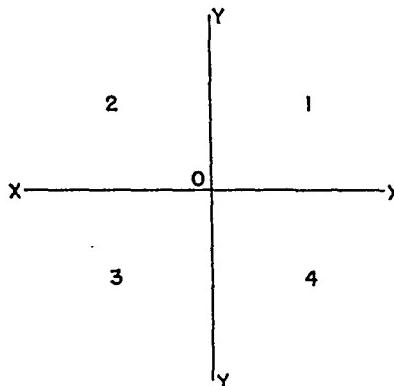
Graphical Solution

There is no elementary algebraic method of solving equations of the third or higher degree unless some of the roots may be discovered by trial and error or by factoring, as explained in the paragraphs bearing these titles. The application of either of these methods from a practical viewpoint is limited, and altogether impossible, in the case of factoring, if irrational roots are involved. However, any equation in any degree, as

long as only one unknown is involved, may have its real roots approximated by a simple although somewhat tedious graphical method. The term real root as introduced here is the same as the term used before simply as root. The term *real* differentiates such roots from *imaginary* roots. By an imaginary root or an imaginary number is meant a number involving the square root, or any even root, of a negative number. For ordinary mathematical purposes it is assumed that the square root of a negative number does not exist. (For actual facts, investigate complex numbers in advanced algebra.) If any of the roots of an equation are imaginary, they will not appear on a graph plotted of that equation.

All algebraic equations can be represented by a line or a curve plotted on a plane. Such a line or curve is established by first finding a number of points, each of which is known to be one point through which the curve of the equation must pass. A number of such points are plotted and through these the curve or line may be drawn which is called the curve or the graph of the equation.

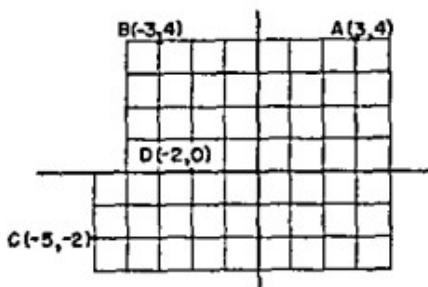
In graphical representation, the plane upon which a point may be located is divided into four zones, or quadrants, which are numbered in a counter-clockwise direction with the upper right-hand quadrant as number one.



The lines separating the quadrants are termed *axes*; the vertical line is designated as the *Y-Y* axis, and the horizontal line as the *X-X* axis. The point of intersection of these two axes is termed the *origin*, or zero point, and from this point all measurements of distance are made. Distance along, or parallel to, the *X-X* axis is termed the *abscissa*, and is considered positive if measured to the right of the *Y-Y* axis and negative if measured to the left. Distance along, or parallel to, the *Y-Y* axis is termed the *ordinate*, and is positive if measured above the *X-X* axis, and negative if measured below.

If a point lies in a plane, its location may be determined by two measurements of distance, or coordinates, one showing its distance from the *Y-Y* axis, (*abscissa*), and the other its distance from the *X-X* axis, (*ordinate*). It is customary to write the coordinates of a point in parenthesis with the *abscissa* first. Thus the plotting of $A(3,4)$, $B(-3,4)$, $C(-5,-2)$, and $D(-2,0)$ places the points in the positions and quadrants as shown below.

SOLUTION OF EQUATIONS



The above system of locating a point on a plane is known as Cartesian coordinates. A second method of locating a point in a plane is described in Section III—Trigonometry. The graphical solution of an equation of the type named above consists of arbitrarily assigning numerical values to the unknown and computing the corresponding values of the equation. The values assumed for the unknown are plotted as abscissas and the corresponding values of the equation as ordinates.

Each pair of values so determined establishes a point and through a number of such points a smooth curve can be drawn. The intersection of this curve with the X-X axis establishes a value of the abscissa, which is a root of the given equation.

In solving equations by the graphical method, the term *function* is used to replace the word equation, and the term *variable* is used in place of unknown. This terminology is appropriate inasmuch as the equation is a function of the unknown since it depends on this latter quantity for its value. To find the value of an equation it is necessary to transpose all of its terms to one side of the equal sign. Consequently, the term function implies an equation in which all terms are arranged in this manner. In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of a value for the unknown as it is the study of the changes of a variable quantity. Thus the terms function and variable more clearly express the relationship of the unknown to the entire equation than the words formerly employed.

The foregoing explanation may be condensed into several distinct operations and thus clarify the actual procedure to be followed.

Step 1.

Arrange all terms of the given equation on one side of the equal sign and simplify the resulting expression, now termed the function.

Step 2

Arbitrarily assign different values for the variable and determine the corresponding values of the function. Plot a number of points corresponding to the number of pairs of values determined in this manner, using the value of the variable as abscissa and the corresponding value of the function as ordinate. Obviously, if the value of the function becomes zero with some assumed value for the variable, then that value of the variable is a root of the equation.

Step 3.

Through the number of points thus established draw a continuous smooth curve. The value of the abscissa of any point at which this curve

either crosses or is tangent to the X-X axis is the value of a root of the given equation. This fact may be verified by substituting the value found into the given equation.

The graphical method of solving any equation in any degree as long as only one unknown is involved, is nothing more than a pictorial representation of the trial and error solution in which the value of the equation, with all terms on one side of the equal sign, becomes zero upon assuming a value of the unknown corresponding to a root.

The examples on page 42 illustrate the method of graphical solution. The equation used for the first example is of the second degree and has roots which are small integers. Its roots have been already found by trial and error. In the second example the usefulness of the graphical solution is more apparent. In both examples the total number of possible roots have been found because in each case the curve crosses the X-X axis the number of times which corresponds to the degree of the equation. Consequently no imaginary roots exist for these selected equations.

It might be suggested at this point that a combination of the trial and error method together with the graphical method is a convenient means of finding either fractional or irrational roots. First use the trial and error method to find the approximate or boundary roots between which the precise root is known to exist, and then, by the graphical method, plot only this portion of the curve using several points adjacent to the axis. The greater the number of points employed, the more exact the curve is represented, and hence the more accurate the approximation will be.

In any graphical representation or solution, the scale of the drawing will be the controlling factor, not considering errors in solution, in the accuracy of the result. Obviously the larger the scale of the drawing, the more accurate can be the solution. For this reason two alternatives are employed to effect more accurate results.

One of these alternatives was applied in the solution of example 2, and consists of using different scales for ordinate and abscissa in order that it may be easier to accurately estimate the point at which the curve crosses the X-X axis.

The second alternative is to plot a second curve, or set of curves, involving only those portions of the original curve which cross the X-X axis. These portion-curves should be drawn through several points located adjacent to the X-X axis, and to a scale several times as large as that of the original continuous curve from which the approximate points of intersection were discovered.

The application of these two expedients introduces no operation other than those already explained.

There are several characteristics of the curves of equations which should be kept in mind when solving such equations by the method of plotting successive points. These are not peculiarities of all equations, but must be recognized when encountered.

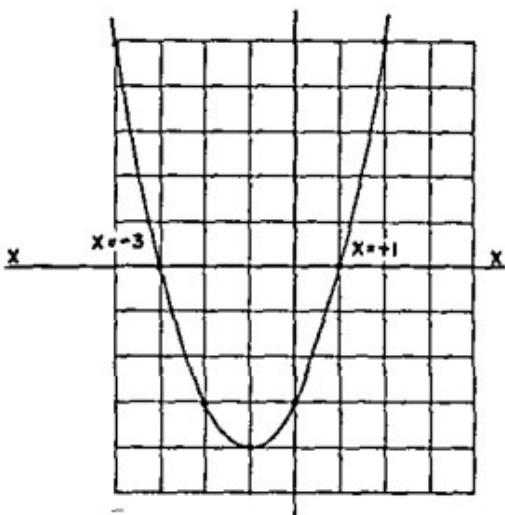
The principle involved in graphically solving algebraic equations is that if the curve of the equation lies above the X-X axis for one value of the unknown and below for another value, there must be a root of the equation somewhere in between. This method is very effective when used in conjunction with certain methods of differential calculus. However, when the only means of obtaining the curve is to plot successive points, the procedure is sometimes very misleading. This is true because it is not always possible to determine the exact shape of the curve near the critical values.

Not demonstrated in the previous examples is the occurrence of *multiple roots* de-

SOLUTION OF EQUATIONS

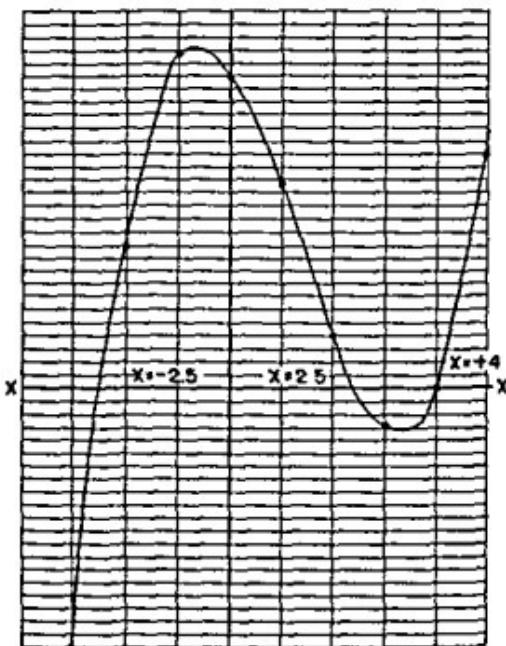
Example 1: $5x^2 + 10x - 15 = 0$
 or $x^2 + 2x - 3 = 0$ (simplified)

LET $X =$	VALUE OF FUNCTION
-5	+12
-4	+5
-3	0
-2	-3
-1	-4
0	-3
1	0
2	+5
3	+12
4	+21



Example 2 $x^3 - 4x^2 - 6.25x + 25 = 0$

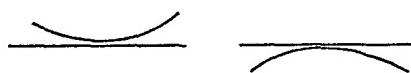
LET $X =$	VALUE OF FUNCTION
-3	-19.75
-2	+11.00
-1	+26.75
0	+25.00
1	+15.75
2	+4.50
3	-2.75
4	0.00
5	+16.75



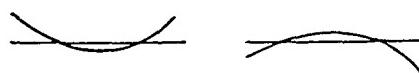
fined as two or more identical roots appearing in the same equation. If any equation containing multiple roots is plotted, the curve will be tangent (touch but not cross) to the X-X axis at a point whose abscissa is the value of the multiple roots. For every such point of tangency it should be considered that two roots of the equation have been determined.

Another factor of importance concerns the plotting of functions in which one or more fractions exist with the variable included in the denominators of one or more of such terms. In such cases the curve will not be continuous, but will break if such values of the variable are assumed which will make the denominator become zero with the numerator some other value.

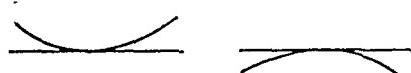
The following sketches illustrate examples where the graphical method of solution either breaks down completely or must be used with great caution.



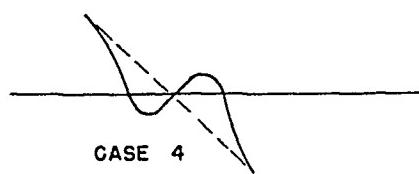
CASE 1



CASE 3



CASE 2

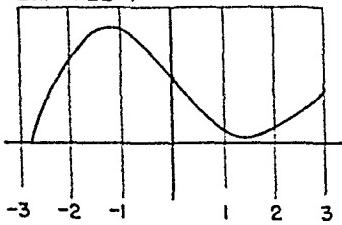


CASE 4

In Case 1 the curve approaches the axis closely, but does not quite touch it. In Case 2 it is tangent to the axis, and in Case 3 it crosses the axis at two near-by points.

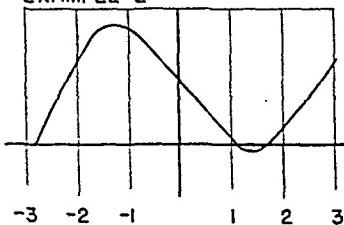
Case 1 can be detected only by advanced methods such as are used in solving the equation of Example 1 shown below. Similarly, Case 2 can be detected only by advanced methods, unless the point of tangency is located by guessing the exact value for the variable. This is unlikely unless some of the roots are small integers. The condition represented by Case 3 can be detected by plotting points sufficiently close together, but is easily overlooked in practice as can be seen in Example 2 below, that is, it may be mistaken for Case 1, and two roots thereby lost. In Case 4 the dotted curve is easily mistaken for the true one so that only one root is believed to exist when actually there are three. This can also be avoided by plotting points sufficiently close together.

EXAMPLE 1



$$3x^3 - 18x + 17 = 0$$

EXAMPLE 2



$$3x^3 - 18x + 17 = 0$$

The bend points of both graphs can be shown to be at $x = -\sqrt{2}$ and $x = +\sqrt{2}$. The curve in Example 1 misses the axis by approximately .03. The curve in Example 2 crosses the axis between 1.1 and 1.2 and again between 1.6 and 1.7. Since the function

of $3x^2 - 18x + 16$ is positive for both $x = 1$ and $x = 2$, the roots between 1 and 2 might both be lost.

Factoring

Factors have been previously defined as numbers that are multiplied together, and also as quantities which can be exactly divided into the given expression. The process of factoring is a reverse process to that of multiplication, for in factoring one starts with a product and endeavors to find the factors which have been multiplied together to produce the given quantity. When the given quantity has been completely factored, the results obtained are called *prime factors*. Such factors are easily recognized since they are exactly divisible only by themselves or by one. Not all quantities are factorable as they may already be in the form of a prime factor, even though the given quantity may consist of several terms and consequently appear to be a product of two or more simpler terms.

The first step, and one of several operations involved in the method of solving an equation by factoring, consists of transposing all terms to one side of the equal sign. The terms are at the same time arranged so that they are in an order in which the factors may be discovered by inspection in some simple cases, or by some method of computation for more complex problems as explained later. Each factor thus found is further factored, if possible, so that the prime factors are obtained. The prime factors are then written in place of the given equation producing an *equivalent* though different appearing relationship. For example in the equation $x^2 + 2x - 3 = 0$, the factors are $(x + 3)$ and $(x - 1)$ as shown on page 42. The partial solution of the equation is.

$$\begin{aligned}x^2 + 2x - 3 &= 0 \\(x + 3)(x - 1) &= 0\end{aligned}$$

An analysis of the equation as represented by the product of its prime factors shows that one factor multiplied by another factor can produce zero as a product only (a) when one of the factors itself is zero, or (b) when both of the factors are zero. This fact provides a means of solving the equation, because if:

$$\begin{aligned}x + 3 &= 0, \text{ then } x = -3, \text{ and if} \\x - 1 &= 0, \text{ then } x = +1.\end{aligned}$$

Thus the roots of the equation are $x = -3$ and $x = +1$. The validity of these roots is established by substituting them into the given equation.

$$\begin{array}{rcl}(-3)^2 + 2(-3) - 3 &=& 0 \\ 9 & - 6 & - 3 = 0 \\ 0 & = & 0 & \text{Check} \\ 1 + 2 - 3 &=& 0 \\ 0 & = & 0 & \text{Check}\end{array}$$

It is advisable at this time for the user of this method of solution to check himself in regard to the definitions of the terms factors and roots, as these two words are often confused. In the example above the factors were stated to be $x + 3$ and $x - 1$. The roots were found to be $x = -3$ and $x = +1$ respectively. From these relationships an important rule can be formulated which, for convenience, is stated in the reverse order to that just given.

- (46). For any equation, of any degree, if n is found to be a root of that equation, then $(x - n)$ is a factor.

The prime factors of many algebraic equations of the second degree are found by the trial and error method simply by finding two factors which when multiplied together will produce the given equation as a product. Since the factors of such equations are usually polynomials it is essential to understand a method of multiplying polynomials together with a minimum of effort. The rule is as follows:

- (47). The product of two polynomials is equal to the algebraic sum of the products obtained when all terms of one polynomial are multiplied by each term of the other polynomial.

The application of this rule is demonstrated by two equivalent methods with the recommendation that the first form be employed until experience justifies the use of the second method.

$$\begin{array}{r}
 (3 - 2) (4 + 5) = \\
 \begin{array}{r}
 3 - 2 \\
 4 + 5 \\
 \hline
 (3) (4) - (2) (4)
 \end{array} \\
 \begin{array}{r}
 + (3) (5) - (2) (5) \\
 \hline
 12 - 8 + 15 - 10 = 9
 \end{array}
 \end{array}$$

$$(3 - 2) (4 + 5) = 9$$

$$\begin{array}{r}
 (x + 3) (x - 1) = \\
 \begin{array}{r}
 x + 3 \\
 x - 1 \\
 \hline
 x^2 + 3x
 \end{array} \\
 \begin{array}{r}
 - x - 3 \\
 \hline
 x^2 + 2x - 3
 \end{array}
 \end{array}$$

$$(x + 3) (x - 1) = x^2 + 2x - 3$$

In computing the product of two or more polynomials, the products of terms involving the same variables with the same exponents are listed in columns as shown. This simplifies the operation of collecting terms.

$$\left(\frac{b^2}{3} - \frac{b}{2} - \frac{1}{2} \right) \left(\frac{b^2}{2} + \frac{b}{5} + \frac{1}{2} \right) =$$

To facilitate solution only, and not a necessary operation, *each* polynomial is reduced to *its* least common denominator. Since the denominator of all products will then be 60, they are not written down in each instance. However, this denominator must be included in the answer.

$$\begin{aligned} & \frac{2b^2}{6} - \frac{3b}{6} - \frac{3}{6} \\ & \frac{5b^2}{10} + \frac{2b}{10} + \frac{5}{10} \\ & \underline{10b^4 - 15b^3 - 15b^2} \\ & \quad 4b^3 - 6b^2 - 6b \\ & \quad \underline{10b^2 - 15b - 15} \\ & \underline{10b^4 - 11b^3 - 11b^2 - 21b - 15} \\ \left(\frac{b^2}{3} - \frac{b}{2} - \frac{1}{2} \right) \left(\frac{b^2}{2} + \frac{b}{5} + \frac{1}{2} \right) &= \frac{10b^4 - 11b^3 - 11b^2 - 21b - 15}{60} \end{aligned}$$

The second method of multiplying factors together is the same as that already shown with the elimination of the tabulation of products.

$$(x+3)(x-1) = x^2 - x + 3x - 3 = x^2 + 2x - 3$$

This procedure can be used in any given problem but becomes of less practical value in finding the products of polynomials if each polynomial consists of more than a few terms.

Polynomials involving radical signs are solved as follows:

Example 1.

$$\begin{aligned} (\sqrt{x-1})(\sqrt{x-1}) &= (\sqrt{x-1})^2 = (x-1)^{1/2}(x-1)^{1/2} \\ &= (x-1)^{2/2} = x-1 \end{aligned}$$

Example 2.

$$(1 + \sqrt{x+3})^2 =$$

$$\begin{aligned} & 1 + \sqrt{x+3} \\ & 1 + \sqrt{x+3} \\ & \underline{1 + 1\sqrt{x+3}} \\ & \underline{1\sqrt{x+3} + (x+3)} \\ & 1 + 2\sqrt{x+3} + x+3 \end{aligned}$$

$$\therefore x + 2\sqrt{x+3} + 4$$

Factors of some equations can be found by comparing the given equation with several type forms for which the factors are known. In these equations the letters of the alphabet, (*a*), (*b*), and (*c*), are arbitrary constants corresponding to the numerical coefficients of the actual equation.

1. $\pm ax \pm ay \pm az = a(\pm x \pm y \pm z)$
2. $x^2 - b^2 = (x - b)(x + b)$
3. $x^2 + 2bx + b^2 = (x + b)(x + b) = (x + b)^2$
4. $x^2 - 2bx + b^2 = (x - b)(x - b) = (x - b)^2$
5. $x^2 \pm (b+c)(x) \pm bc =$ (see below)
6. $x^3 + b^3 = (x + b)(x^2 - bx + b^2)$
7. $x^3 - b^3 = (x - b)(x^2 + bx + b^2)$

The forms 3 and 4 listed above are included in type 5. Equations of this type are

termed quadratic equations or equations of the second degree. The equations $x^2 = 16$ and $x^2 - 4x = 0$ are also quadratic, but are excluded from this discussion as these types are more easily solved by rules (43) and (40) respectively. Bearing in mind that the proper sign \pm must be considered, the values to be assigned as the second term in each factor can be determined as follows:

The last term, bc , of the equation is the algebraic product of the second terms of the factors, and, at the same time, the coefficient of the x term is the algebraic sum of the second terms of the factors.

Using the equation $x^2 + 2x - 3 = 0$, from the previous paragraph, and by the application of the rule as just stated, the factors, $x + 3$ and $x - 1$, can be written, since $+3$ and -1 are the only two numbers which will give -3 as a product and $+2$ as their sum.

An equation of the form $ax^2 \pm bx \pm c = 0$ can be factored by the same method as above if the equation is first divided through by the coefficient of the x^2 term, represented in the equation by (a). However, in some cases it is more convenient to factor these equations directly from the form in which they are given by the application of certain apparent relationships.

$$\begin{aligned} 2x^2 + 7x + 3 &= 0 \\ (2x + 1)(x + 3) &= 0 \end{aligned}$$

The product of the first terms of the factors is the first term of the given equation.

The sum obtained by adding the product of the two end terms of the factors to the product of the two nearest terms is the middle term of the given equation.

The products of the last terms of the factors is the third term of the given equation.

$$\begin{array}{ll} (2x + 1)(x + 3) = 0 & \\ 2x + 1 = 0 & x + 3 = 0 \\ 2x = -1 & \\ x = -\frac{1}{2} & x = -3 \end{array}$$

The solution of equations by factoring is an important method and is applicable to equations in any degree provided that such equations are factorable. In practice, however, factoring is subject to limitations because there are so many equations whose factors cannot be found conveniently, if at all. This is true where the roots of the equation are fractional or irrational. However, many equations of the third and higher degree are factorable and the explanation to follow will demonstrate the method of their solution which will include an explanation of the division of polynomials, an operation usually required in the solution of such equations.

In the previously described method for finding the factors of an equation no mention was made of finding the roots of the given equation by trial and error and from these determining what the factors would be according to Rule (46):

In any equation, of any degree, if (n) is found to be a root of that equation, then $(x - n)$ is a factor.

This was purposely omitted in the earlier paragraphs to preclude the possibility of yielding to the temptation of working the problem in the reverse order, that is, deter-

minding the roots and then writing the corresponding factors. However, for equations of higher degree such a procedure is an essentiality in order to determine as many of the roots as possible. For the purpose of explanation, assume that the equation $x^4 + 4x^3 - 5x^2 + 36x - 36 = 0$ is to be factored. By arbitrarily assuming several small integers as values of the variable, one of the roots of the equation is discovered, most likely that $x = -2$. Further application of the trial and error method of solution in this particular example would disclose other roots, but as a general procedure, it is advisable to simplify, or factor the given equation after each root is discovered. Since $x = -2$ is known to be a root of the equation, then $(x + 2)$ is a factor, or:

$$(x + 2) \text{ (Product of other factors)} = x^4 + 4x^3 - 5x^2 + 36x - 36$$

The value of the product of the other factors can be determined by dividing the given equation by the known factor $(x + 2)$. This is explained by steps so that the procedure as outlined may serve as an explanation for the division of any polynomial by another.

Step 1.

Arrange the terms of the expression to be divided and the divisor so that the exponents of each are in a descending order. When the division is exact, that is, without remainder, the terms of both dividend and divisor can also be arranged so that the exponents of each are in an ascending order.

$$x + 2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36}$$

Step 2.

Write as the first term in the quotient the result obtained when the first term of the dividend is divided by the first term of the divisor.

$$\begin{array}{r} x^3 \\ x + 2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \end{array}$$

Step 3.

Multiply the divisor by the first term in the quotient obtained in Step 2 and write the product under the dividend placing the terms containing like exponents under the corresponding terms of the dividend.

$$\begin{array}{r} x^3 \\ x + 2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \\ \underline{x^4 + 2x^3} \end{array}$$

Step 4.

Subtract the product obtained in Step 3 from the dividend and to the remainder annex additional terms taken from the dividend so that there are in all a number of terms equal to that of the divisor.

$$\begin{array}{r} x^3 \\ x + 2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \\ \underline{x^4 + 2x^3} \\ \underline{\underline{2x^2 - 5x^2}} \end{array}$$

Step 5.

Write as the second term in the quotient the result obtained when the first term of the expression obtained in Step 4 is divided by the first term of the divisor.

$$\begin{array}{r} x^3 + 2x^2 \\ x+2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \\ x^4 + 2x^3 \\ \hline 2x^3 - 5x^2 \end{array}$$

Step 6.

By an operation similar to Step 3, multiply the divisor by the second term in the quotient obtained in Step 5, and write the product under the dividend, placing the term containing like exponents under the corresponding terms of the dividend.

$$\begin{array}{r} x^3 + 2x^2 \\ x+2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \\ x^4 + 2x^3 \\ \hline 2x^3 - 5x^2 \\ 2x^3 + 4x^2 \\ \hline \end{array}$$

Step 7.

Continue as in Steps 4, 5, and 6 until all terms of the dividend have been used. In the above example this requires six more operations, and the complete solution appears as below:

$$\begin{array}{r} x^3 + 2x^2 - 9x - 18 \\ x+2 \overline{)x^4 + 4x^3 - 5x^2 - 36x - 36} \\ x^4 + 2x^3 \\ \hline 2x^3 - 5x^2 \\ 2x^3 + 4x^2 \\ \hline - 9x^2 - 36x \\ - 9x^2 - 18x \\ \hline - 18x - 36 \\ - 18x - 36 \\ \hline \end{array}$$

Therefore: $(x+2)(x^3 + 2x^2 - 9x - 18) = 0$

A comparison of the quotient $x^3 + 2x^2 - 9x - 18$ with the type forms of cubic equations which are factorable by inspection indicates that an expression of this form cannot be factored by this method. Any further factoring must be accomplished by other means. The trial and error method may be employed to find a second root of the equation by finding some value of the variable which will satisfy either the original equation or the expression remaining to be factored. The same numerical value will satisfy both of these. Consequently, the factor having the fewer number of terms should be investigated.

An investigation of the expression $x^3 + 2x^2 - 9x - 18$ shows that $x = 3$ is a root and $(x - 3)$ is a factor of this expression and, therefore, of the original equation as

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well. Consequently, a choice exists between dividing the original equation by the product of $(x + 2)$ and $(x - 3)$ or to divide the expression $x^3 + 2x^2 - 9x - 18$ by $(x - 3)$. The latter procedure is adopted and is accomplished by the method previously described for the division of one polynomial by another. The result obtained from this division is $x^2 + 5x + 6$.

$$\text{Therefore, } (x + 2)(x - 3)(x^2 + 5x + 6) = 0$$

The expression $x^2 + 5x + 6$ is a quadratic of the type $x^2 + bx + c = 0$ and for the values $b = 5$ and $c = 6$, the prime factors $x + 2$ and $x + 3$ are obtained.

$$\text{Thus, } (x + 2)(x - 3)(x + 3)(x + 2) = 0$$

$$x = -2; +3; -3; -2$$

The presence of four integral roots as the result of the above somewhat laborious process might lead to the conclusion that such equations should be completely solved by the method of trial and error. This may be true for the example as given, but such convenient equations are not likely to occur often in any actual problem. In this instance the equation was purposely constructed so as to have integers for roots suitable for a rapid solution. Furthermore, the existence of multiple roots as in this example defies a solution by trial and error. Later examples in the paragraphs titled Equations of Higher Degree Solved as Quadratics further justifies the factoring method of solution for such equations.

In the solution of equations by the factoring method, the use of a factor having a numerical value of zero results in either an erroneous or an incomplete answer. Rule (40) states that both sides of an equation may be divided by the same or by an equal quantity provided that quantity is not zero. This qualification, that the divisor must not be zero seems illogical when it is considered that this same quantity is so frequently used as a factor in the reverse operation of multiplication. However, it is easy to realize that any number taken zero times is still zero, or

$$(50)(0) = 0$$

But if the comparable equation involving division is examined,

$$\frac{50}{0} =$$

the answer to be assigned as a quotient is highly imaginative as no number can be written as a quotient, which, when multiplied by the divisor (0) will again produce the number 50. Thus it seems best to state it as a fact that zero cannot be employed as a divisor, rather than to say that the quotient is some indefinite quantity such as infinity.

The fallacy of dividing by zero is apparent in the examples below.

Given,

$$a = b$$

$$a^2 = ab$$

$$a^2 - b^2 = ab - b^2$$

$$(a - b)(a + b) = b(a - b) \text{ Dividing by } (a - b)$$

$$a + b = b$$

which is 0

But $a = b$, so,

$$2b = b$$

$$2 = 1$$

Given, $(2x + 1)(x + 3) = x^2 - 9$
 $(2x + 1)(x + 3) = (x - 3)(x + 3)$
 $2x + 1 = x - 3 \quad \text{Dividing by } (x + 3)$
 $x = -4$

An inspection of the original equation shows that it is of the second degree and consequently has two roots.

Since only one root results from the solution as performed, it is possible that the quantity $(x + 3)$ used as a divisor may be zero, and hence the operation incorrect. If the divisor $(x + 3)$ is equal to zero, then (x) is equal to -3 . Substituting this value into the equation proves $x = -3$, a root of the equation. Consequently the divisor $(x + 3)$ was equal to zero, and the erroneous operation performed resulted in the loss of one of the roots.

It is recommended that before any equation is ever simplified by dividing each term of that equation by a common factor involving the variable, the value of that factor first be proved to be other than zero. This is easily done as described in the previous paragraph. If the value of the factor is not zero, then it may be used as a divisor according to Rule (40).

Completing the Square

The title of this method of solution results from the fact that all terms of the equation involving the variable are transposed to one side of the equal sign after which an appropriate constant term is added so that this side of the equations is arranged to form a perfect square. A *perfect square* may be defined as a quantity which has an exact square root. The other side of the equation consists of the remaining portion of the original equation together with the same constant term as added to complete the square on the side of the equation involving the variables. After getting the equations into the form described, the next operation is to take the square root of both sides simultaneously, and then simplify the resulting equation.

The method of completing the square is used almost altogether in the solution of quadratic equations, although higher equations which can be arranged in the form of perfect squares can be similarly solved.

$$x^2 - 3x + 2 = 0$$

B 23 H 3

The equation as given is obviously not in the form of a perfect square. To change it into such form the constant term 2 must be added to or subtracted from according to whether its given value is less or greater than the desired value. The other side of the equation must of course be equally affected, Rule (36). A simpler solution is to transpose the given constant term to the right-hand side of the equation and then determine what value is required for the constant to make the left-hand side a perfect square. Once this is determined, the appropriate number is added and at the same time an equal quantity is added to the other side. Thus the equality of the relationship is unimpaired.

Whenever the coefficient of the x^2 term is equal to one (as in this case), the value of the constant term is necessary to make the left-hand side of the equation a perfect square is equal to the *square* of one-half of the coefficient of the x term. Note that it is the square of one-half of the *coefficient* of the x term and has nothing to do with the variable x .

SOLUTION OF EQUATIONS

$$x^2 - 3x = -2$$

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = -2 + \frac{9}{4} = -\frac{8}{4} + \frac{9}{4} = \frac{1}{4}$$

$$\left(x - \frac{3}{2}\right) \left(x - \frac{3}{2}\right) \text{ or } \left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

These last two equations are equivalent since the square of the factor $(x - 3/2)$ is equal to the left-hand side of the equation above. The two right-hand sides of the equation are identical. To write the left-hand side of the equation as a factor $(x - 3/2)$ directly from its equivalent expression in the equation above note these facts

1. The first term of the factor is the square root of the first term of the equivalent expression.
2. The second term of the factor is the square root of the last term of the equivalent expression.
3. The sign \pm of the factor is the *same* as the sign of the middle term of the equivalent expression.

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}}$$

$$x - \frac{3}{2} = \pm \frac{1}{2}$$

$$x = \frac{3}{2} \pm \frac{1}{2}$$

$$x = \frac{3}{2} + \frac{1}{2} = 2$$

$$x = \frac{3}{2} - \frac{1}{2} = 1$$

Not all quadratic equations encountered will have coefficients of their x^2 terms equal to one, even though the equation is completely simplified. Nevertheless this coefficient can always be made equal to this value by dividing each term of the equation by the coefficient of the x^2 term if its given value is greater than one, or by multiplying by its reciprocal if its value is fractional.

$$4x^2 - 9x + 2 = 0$$

$$x^2 - \frac{9}{4}x = -\frac{2}{4}$$

$$x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 = -\frac{2}{4} + \left(\frac{9}{8}\right)^2 = -\frac{32}{64} + \frac{81}{64} = \frac{49}{64}$$

$$\left(x - \frac{9}{8}\right)^2 = \frac{49}{64}$$

$$\sqrt{\left(x - \frac{9}{8}\right)^2} = \sqrt{\frac{49}{64}}$$

$$x - \frac{9}{8} = \pm \frac{7}{8}$$

$$x = \pm \frac{7}{8} + \frac{9}{8}$$

$$x = +2; x = \frac{1}{4}$$

The operation of changing the equation so that the coefficient of the x^2 term is equal to one is not a necessary operation in all cases. The left hand side of the equation can be arranged in the form of a perfect square by principles explained in the paragraph entitled Factoring, for the arrangement of a polynomial in the form of a perfect square is nothing more than the finding of two identical factors of that polynomial. This procedure is demonstrated in the following example which also shows the convenience and advisability of using the method of completing the square for equations involving irrational roots.

$$5r^2 + 7r - 2 = 0$$

$$100r^2 + 140r = 40$$

$$100r^2 + 140r + 49 = 40 + 49 = 89$$

$$(10r + 7)(10r + 7) = 89$$

$$(10r + 7)^2 = 89$$

$$10r + 7 = \sqrt{89} = \pm 9.44$$

$$r = \pm \frac{9.44}{10} - \frac{7}{10} = \pm .944 - .7$$

$$r = .244; -1.644$$

Quadratic Formula

Quadratic equations of the form $ax^2 \pm bx \pm c = 0$ and $x^2 \pm bx \pm c = 0$ are so frequently encountered that a formula has been derived to show the value of the variable (x) in terms of the coefficients (a) and (b) and the constant term (c). For simplicity, the general equation $ax^2 + bx + c = 0$ with all of its terms positive is chosen to represent any quadratic equation. From this equation the value of (x) is determined by the method of completing the square. The result of the solution is in the form of an equation, but is more commonly referred to as a formula. By substituting from an equation with numerical values for coefficients and constant terms instead of (a),

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(b), and (c), the numerical value of the roots of the equation may readily be determined.

$$ax^2 + bx + c = 0$$

$\left(\begin{array}{l} \text{Dividing equation by } a \\ \text{and transposing } c \end{array} \right)$	$x^2 + \frac{bx}{a} = -\frac{c}{a}$
$\left(\begin{array}{l} \text{Adding } \left(\frac{b}{2a}\right)^2 \text{ to} \\ \text{both sides of equation} \end{array} \right)$	$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$
$\left(\begin{array}{l} \text{Factoring left side} \\ \text{of equation.} \end{array} \right)$	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
$\left(\begin{array}{l} \text{Taking square root} \\ \text{of both sides.} \end{array} \right)$	$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
$\left(\begin{array}{l} \text{Transposing and} \\ \text{simplifying} \end{array} \right)$	$x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

This formula is valid only for equations which are similar to the general equation from which it was derived. The specifications for such an equation are as follows:

- 1 There must be only one variable in the equation to be solved. Since (x^2) and (x) are the same variable raised to different powers, there is only one variable present in the general equation as stated above.
- 2 There may be any number of constant terms, (d), (e), (f), (g), and so on, on either side of the equation as assigned, but before solution by the formula all such terms must be transposed and combined into one constant term (c).
- 3 The sign (\pm) of the terms need not be all positive (+) as in the general equation, $ax^2 + bx + c = 0$, but the sign of the terms must be correctly considered when substituting in the formula.

The formula as derived should be memorized because it is the most widely used method of solution for quadratic equations. It is of special convenience, compared to other methods, for solving equations involving either fractional, or irrational roots. This is evident by an inspection of the second example.

Since the type of equation solved by this method is of the second degree, there will be two roots from each solution. This fact is indicated by the presence of the \pm sign preceding the radical sign ($\sqrt{}$). The quantity $b^2 - 4ac$ beneath the radical sign is termed the *discriminate* because its value determines the nature of the roots.

If $b^2 - 4ac > 0$, the roots are real and unequal.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac < 0$, the roots are imaginary.

With the possible exception of the last, these three facts seem obvious. If the value of the expression ($b^2 - 4ac$) is less than zero, that is, it is negative, then the roots are imaginary since the square root of a negative number cannot be represented by real numbers, either positive or negative.

$$\begin{aligned}5x^2 + 10x + 15 &= 30 \\5x^2 + 10x - 15 &= 0\end{aligned}$$

Comparing this with the general form,

$$ax^2 + bx + c = 0,$$

It is obvious that in the equation to be solved

$$\begin{aligned}a &= +5, \\b &= +10, \\c &= -15.\end{aligned}$$

Now, substituting this in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-10 \pm \sqrt{100 - 4(5)(-15)}}{10}$$

$$x = \frac{-10 \pm \sqrt{400}}{10}$$

$$x = \frac{-10 \pm 20}{10}$$

$$x = -3$$

$$x = +1$$

The magnitude of the numbers corresponding to (a), (b), and (c) substituted in the formula for the solution of the above example could have been reduced to smaller integers by dividing the given equation by the coefficient of the (x^2) term. The results obtained in either case are identical because the two equations are equivalent.

$$\begin{aligned}x^2 + 2x - 3 &= 0 \\a = 1; b = 2; c = -3\end{aligned}$$

Such simplification as suggested above is advisable in many cases provided that the values for (b) and (c) remain integers. Any fractions resulting from such a division would most likely increase and not decrease the labor involved in the application of the formula.

$$\begin{aligned}x^2 - 8x + 8 &= 0 \\a = 1; b = -8; c = 8\end{aligned}$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(8)}}{2}$$

$$x = \frac{8 \pm \sqrt{64 - 32}}{2} = \frac{8 \pm \sqrt{32}}{2}$$

$$x = \frac{8 \pm \sqrt{(16)(2)}}{2} = \frac{8 \pm 4\sqrt{2}}{2}$$

$$x = 4 \pm 2\sqrt{2}$$

$$x = 4 \pm 2(1.414) = 4 \pm 2.828$$

$$x = 6.828$$

$$x = 1.172$$

If the (a) term or coefficient of (x^2) is not positive as it appears in the equation to be solved, it is advisable to make it positive as a first step in the solution. This is accomplished by multiplying the entire equation by (-1) , which is equivalent to changing to the reverse sign each and every term of the equation.

The greatest source of error in solving equations by the use of this method is the failure to correctly apply the correct sign (\pm) for the various terms when substituting in the formula. For instance in the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the first term to be substituted is $(-b)$. If the number corresponding to (b) , which is the coefficient of the (x) term in the equation to be solved, is already negative, then it becomes positive $(+)$ when it is substituted in the equation. This is true, since $-(-b) = +b$. The (b^2) term is always positive, since the square of any number, either positive or negative, is always positive. In all cases the number corresponding to the (a) term, which is the coefficient of (x^2) , will be positive, or can be made positive as already explained. In such cases, the expression, $-4ac$, will be minus or plus, respectively, as (c) is plus or minus. The term, $(2a)$, in the above case would always be positive.

Errors also frequently arise from dividing only part of the numerator by the term $2a$. As shown by the formula, the entire numerator is to be divided, and this is certain to be done if care is taken to extend the divisional bar from the equal sign to the extreme right-hand side of the fraction.

The formula method of solution is not limited in its application to quadratic equations.

However, any equation solved by use of the formula must be of the same type as a quadratic, that is with the variable of the first term the square of the variable of the second term. Examples of this type are shown in the paragraphs entitled Equations of Higher Degree Solved as Quadratics.

Radical Equations

Equations in which the unknown quantity appears under a radical sign are called *radical equations*. The variable may also appear elsewhere in the equation without the radical sign, and in some cases two or more radical signs are involved in the same equation, each containing the variable.

The solution of a radical equation requires that all radical signs be removed. This is accomplished by isolating such terms on one side of the equal sign and then squaring both sides of the equation. If the equation contains more than one radical term, transposing and squaring a second or third time may be necessary.

This squaring of the terms on both sides of the equal sign raises the degree of the equation. As a result additional roots may be formed which will not check when substituted into the given equation. The operation of checking is therefore a necessary part of the solution in this type of problem. Any results which do not check in the original equation are termed *extraneous roots* and are consequently eliminated.

$$\begin{aligned}
 2x + 4 &= 6 + \sqrt{x+2} \\
 2x + 4 - 6 &= \sqrt{x+2} \\
 2x - 2 &= \sqrt{x+2} \\
 (2x-2)^2 &= x+2 \\
 4x^2 - 8x + 4 &= x+2 \\
 4x^2 - 9x + 2 &= 0 \\
 x &= +2 \\
 x &= +0.25
 \end{aligned}$$

The original equation is of only the *first* degree, and hence has only *one* root. Since two unidentical values for the root have been found, one of them is obviously false. The correct root may be determined by substituting one of the values in the original equation, and if it satisfies, then it is the true root of the equation. If it does not satisfy, then the remaining value is the correct root. This may be proved by substituting it into the equation. In this example, 2 is found to be the true root of the equation.

If the equation contains more than one radical, it may be necessary to remove one of these radicals at a time by isolating it on one side of the equal sign and then squaring both sides of the equation. A repetition of this process will eventually eliminate all radical terms.

$$\begin{aligned}
 \sqrt{3x-2} - \sqrt{x+3} &= 1 \\
 \sqrt{3x-2} &= 1 + \sqrt{x+3} \\
 3x - 2 &= 1 + 2\sqrt{x+3} + x + 3 && \text{Rule (42)} \\
 2x - 6 &= 2\sqrt{x+3} \\
 x - 3 &= \sqrt{x+3} \\
 x^2 - 6x + 9 &= x + 3 \\
 x^2 - 7x + 6 &= 0 \\
 x &= 6 \quad (\text{true root}) \\
 x &= 1 \quad (\text{extraneous root})
 \end{aligned}$$

Equations of Higher Degree Solved as Quadratics

Many equations of the third or higher degree may be solved without resort to the graphical solution if the given equation can be represented as the product of several factors, or as an equation of lesser degree. Some of the possibilities of such solutions are indicated by the few examples which follow:

$$\begin{aligned}
 \frac{144}{y^2} + y^2 &= 25 \\
 144 + y^4 &= 25y^2 \\
 y^4 - 25y^2 + 144 &= 0 \\
 (y^2 - 16)(y^2 - 9) &= 0 \\
 y^2 &= 16 & y^2 &= 9 \\
 y &= \pm 4 & y &= \pm 3
 \end{aligned}$$

An alternative solution may be performed by letting $y^2 = x$, thus making the given equation a simple quadratic. The formula may then be employed.

SOLUTION OF EQUATIONS

$$a = 1; b = -25; c = 144$$

$$y^2 = \frac{25 \pm \sqrt{625 - (4)(144)}}{2} = \frac{25 \pm \sqrt{49}}{2}$$

$$\begin{aligned}y &= \pm 4 \\y &= \pm 3\end{aligned}$$

Equations of the first degree may also be solved as quadratics by the same principle that applies to equations of higher degree as in the previous example.

$$\begin{aligned}2x + 4 &= 6 + \sqrt{x+2} \\2(x+2) - \sqrt{x+2} - 6 &= 0 \\2(\sqrt{x+2})^2 - \sqrt{x+2} - 6 &= 0 \\a = 2, b = -1; c = -6\end{aligned}$$

$$\sqrt{x+2} = \frac{1 \pm \sqrt{1 - 4(2)(-6)}}{2(2)} = \frac{1 \pm \sqrt{49}}{4} = \frac{1}{4} \mp \frac{7}{4}$$

$$\sqrt{x+2} = \frac{8}{4} = 2 \quad \sqrt{x+2} = \frac{-6}{4} = -\frac{3}{2}$$

$$x+2 = 4 \quad x+2 = \frac{9}{4}$$

$$x = 2 \quad x = 1/4$$

These roots should be checked in the original equation. In this example this operation shows $x = 25$ to be an extraneous root. (See Radical Equations).

$$x^6 - 19x^3 - 216 = 0$$

This is really a quadratic in terms of x^3 , for if some letter such as (z) is substituted for (x^3), the equation becomes

$$z^2 - 19z - 216 = 0$$

the roots of which are $z = 27$, and $z = -8$

There are in all, six roots for the original equation, but only two are real as shown below.

$$\begin{array}{ll} z = x^3 = 27 & z = x^3 = -8 \\ x^3 - 27 = 0 & x^3 + 8 = 0 \\ \text{Factoring: } (x-3)(x^2+3x+9) = 0 & (x+2)(x^2-2x+4) = 0 \\ \text{Roots: } x-3 = 0 & x+2 = 0 \\ & (b^2 - 4ac = -27) \\ & x^2 - 2x + 4 = 0 \quad (b^2 - 4ac = -12) \end{array}$$

The discriminates of the two quadratic equations are negative. Their roots are imaginary. The only real roots are $x = 3$ and $x = -2$.

In some instances an equation of the third or higher degree can be factored into two or more polynomials, one or more of which is a quadratic. Each of the factors is then solved for the values of the roots. This is demonstrated by an example on page 47.

Another example is given below. In the solution it is interesting to note the application of several of the methods of solution which have been described previously.

$$4x^4 - 9x^3 - 21x^2 + 41x - 15$$

By trial and error $x = 1$ and $x = 3$ are found to be roots of the equation. Therefore $(x - 1)$ and $(x - 3)$ are factors of the equation.

$$\begin{array}{r} (x - 1)(x - 3)(\quad) = 0 = 4x^4 - 9x^3 - 21x^2 + 41x - 15 \\ (x^2 - 4x + 3)(\quad) = 4x^4 - 9x^3 - 21x^2 + 41x - 15 \\ \hline 4x^2 + 7x - 5 \\ x^2 - 4x + 3 \mid 4x^4 - 9x^3 - 21x^2 + 41x - 15 \\ \hline 4x^4 - 16x^3 + 12x^2 \\ \hline 7x^3 - 33x^2 + 41x \\ 7x^3 - 28x^2 + 21x \\ \hline - 5x^2 + 20x - 15 \\ - 5x^2 + 20x - 15 \\ \hline \end{array}$$

The quotient as found is a prime factor of the given equation. The irrational roots of this quadratic can be found by either completing the square, or by the use of the quadratic formula.

$$\begin{aligned} (x - 1)(x - 3)(4x^2 + 7x - 5) &= 0 \\ 4x^2 + 7x - 5 &= 0 \\ x &= 0.54\dots \\ x &= -2.3\dots \end{aligned}$$

The roots of the given equation are therefore:

$$x = 1; x = 3; x = 0.54\dots; \text{ and } x = -2.3\dots$$

The same quotient $(4x^2 + 7x - 5)$ could have been obtained by dividing the given expression $(4x^4 - 9x^3\dots \text{etc.})$ by the binomial $(x - 1)$ and the quotient thus obtained divided in turn by the binomial $(x - 3)$. The terms binomial and trinomial are used to designate polynomials of two and three terms, respectively.

METHODS OF SOLUTION—SIMULTANEOUS EQUATIONS

The previously outlined methods of solution are applicable to equations containing one variable or unknown, such as $x + 4 = 6$ and $ax^2 \pm bx \pm c = 0$. It must be clearly understood that although this latter equation contains both x and x^2 , there is but one unknown in the equation, because x and x^2 , are merely different forms of the same unknown, and the finding of one leads directly to the value of the other by a process of substitution. But if the equation contains two or more variables, as in the equation $x^2 + xy + 4 = 3$, then there must be as many independent equations written as there are unknowns appearing in the equations. Each variable is counted only once regardless of the number of times it appears. The simultaneous solution of these equations will result in sets of values and not as separate values for x and for y , a fact made clear by the examples included in the paragraphs describing graphical solutions.

When two equations involving two unknowns are solved simultaneously, the num-

ber of pairs of values obtainable can never be greater, and may be less, than the *product* of the degrees of the equations. The degree of any equation is determined by the term with the highest degree. By the degree of a term is meant the sum of the exponents of that term. Thus the terms x^3 , xy^2 , x_2y , are all of the third degree. The equations $x^2 + y^2 = 25$, and $xy = 12$ are both of the second degree.

The simultaneous solution of two or more equations, so that the sets of values obtained will satisfy all of the equations involved can be accomplished by one or more of the several methods now to be described.

Graphical Solutions

The solution of two equations involving two unknowns is given by their point or points of intersection when both equations are plotted to the same scale on coordinate paper. In a single equation of two, and only two, variables, it is impossible to find the value of either one of them unless the value of the other is known. However, if some arbitrary numerical value is assigned to one of these, the equation can be solved for the corresponding value of the other variable. The variable to which the arbitrary value is assigned is called the *independent variable*, and the remaining variable, since it is a function of the independent quantity, becomes the *dependent variable*. In an equation involving (x) and (y) , either of the two may be chosen as the independent variable, and the remaining variable consequently becomes dependent. The value of the independent variable is customarily plotted as abscissa and the dependent variable as ordinate. This order may be reversed at any time, even when plotting successive points in the same equation. If the independent variable is represented by (x) and the corresponding value of the dependent variable by (y) , a point, P , with coordinates (x) and (y) is established. A number of such points may be similarly plotted and through these points a continuous curve or line may be drawn.

The second equation is similarly plotted and if the curves of the two equations intersect, the abscissa and the ordinate of the points of intersection are the values of (x) and (y) respectively, of these two variables in either equation. The maximum number of points of intersection of two equations is governed by the product of the degrees of the equation as previously explained. This maximum number of intersections is not always realized because of imaginary roots.

An equation of the general form $ax \pm by \pm c = 0$ is called a linear equation since all its plotted points will fall on a straight line. This will always be the case where both of the variables appear separately and with exponents equal to one, but not as their product xy , or their quotient x/y . Since a straight line is determined by knowing two points on it, the graph of a linear equation can be drawn when only two points have been plotted. Usually, but not necessarily always, the most convenient points to choose are those located where the line crosses the two axes. These two points are found by assuming $x = 0$ and finding (y) , and then letting $y = 0$ and finding (x) . These two values thus found for (x) and (y) are called *intercepts* as the line crosses the axes at these points.

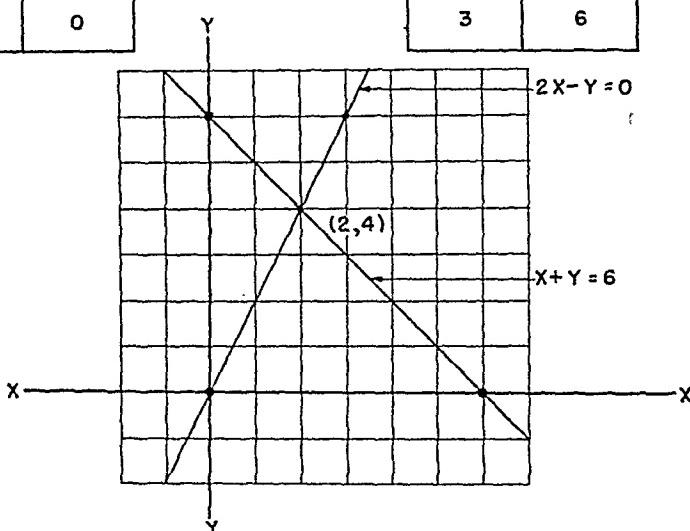
In some cases the X-intercept and the Y-intercept are both zero and consequently the line passes through the origin. This condition is at once apparent whenever the value of the dependent variable is zero for a corresponding zero value of the independent variable. Straight lines which pass through the origin can be plotted whenever one other pair of values of (x) and (y) are known.

$$(A) \quad x + y = 6 \quad \text{or} \quad x = 6 - y$$

$$(B) \quad 2x - y = 0 \quad \text{or} \quad x = \frac{1}{2}y$$

$x = 6 - y$	
LET $x =$	THEN $y =$
0	6
6	0

$x = \frac{1}{2}y$	
LET $x =$	THEN $y =$
0	0
3	6



The point of intersection is found to be $(2,4)$ which gives $x = 2$ and $y = 4$ as the variables which will satisfy both equations, and is therefore the common solution. This fact can be verified by substituting the found values into the given equations. In this case both values of the variable are positive since the point of intersection lies in the first quadrant.

An inspection of the two given equations shows that each of them is of the first degree. The product of the degrees of the equations is therefore one, and the greatest possible set of answers which can be found is one. This fact is apparent for this example without reference to any algebraic theory because two straight lines can intersect at only one point.

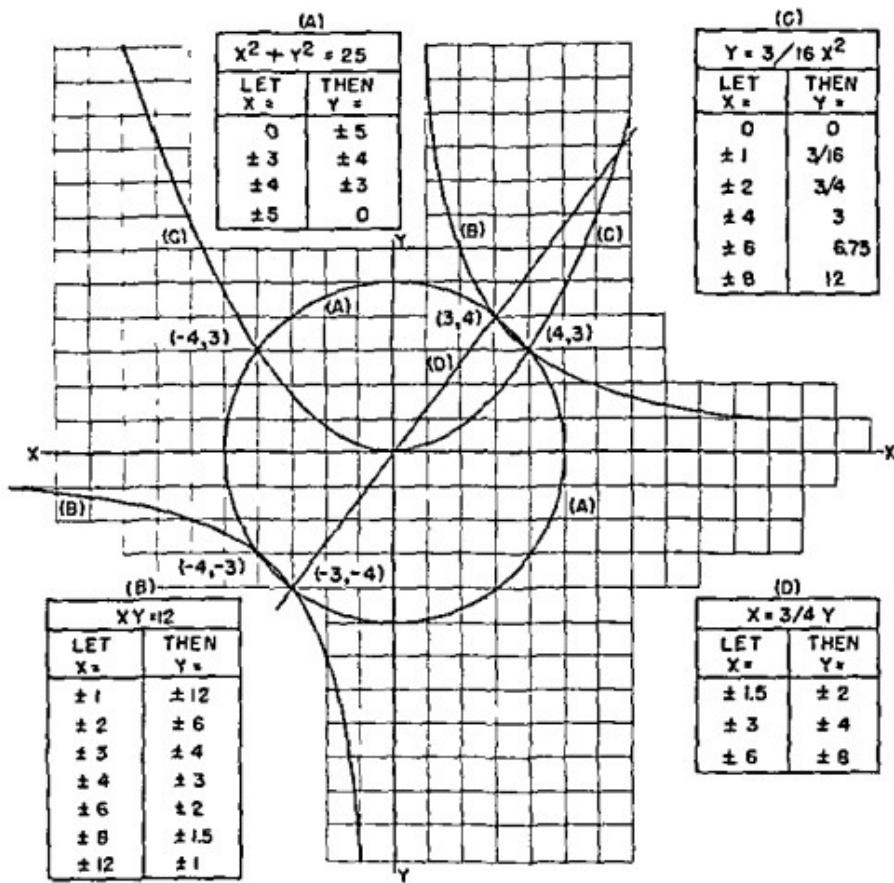
Not all equations plot as straight lines, but as curves. In such cases there may be more than one point of intersection, and hence a corresponding number of sets of values found for the variables. The following three examples are all plotted as one diagram to show the general appearance of the four common types of curves, (A) circle, (B) hyperbola, (C) parabola, (D) straight line. It should be noted that the maximum number of points of intersections are obtained in the simultaneous solution of equations (A) and (B) and of equations (A) and (D). Equations (A) and (C) apparently have two sets of imaginary roots which of course do not appear on the graph.

SOLUTION OF EQUATIONS

(A) $x^2 + y^2 = 25$
 (B) $xy = 12$ or $x = \frac{12}{y}$

(A) $x^2 + y^2 = 25$
 (C) $3x^2 = 16y$

(A) $x^2 + y^2 = 25$
 (D) $4x - 3y = 0$



Generally speaking, the graphical solution of two or more equations is not a satisfactory method from the practical viewpoint. Its chief value lies in its ability to picture to the student the meaning of *common solution* and to indicate the significance of *sets of values* for the variables. These facts are important to the understanding of succeeding methods. Additional information concerning straight lines is included in Section V—Analytical Geometry of Straight Lines.

Substitution

Whenever the value of one of the variables can be determined in terms of another variable, the total number of unknowns can be reduced by one. This is especially useful for solving two equations simultaneously when one of the equations includes variables of the first power only. In such cases the first degree equation is solved for the value of one of the variables in terms of the other variable and any accompanying constant terms which may be involved. This computed value is then substituted in the remain-

ing equation in place of the variable to which it is equivalent. The resulting equation involves but one unknown, and is easily solved by previously described methods.

$$\begin{array}{r} 2x^2 + 2xy + 10 = 50 \\ \underline{x - y = 3} \\ \text{or } x = 3 + y \end{array}$$

$$\begin{aligned} 2(3+y)^2 + 2(3+y)y + 10 &= 50 \\ 2(9+6y+y^2) + 6y + 2y^2 + 10 &= 50 \\ 18 + 12y + 2y^2 + 6y + 2y^2 + 10 &= 50 \\ 4y^2 + 18y - 22 &= 0 \\ y &= 1; -5.5 \\ x &= 4; -2.5 \end{aligned}$$

The substitution method can also be used to solve sets of three or more equations. The method consists of solving any one of the equations for (x) or (y) or (z) and substituting the value found in the other equations. For example, in the three simultaneous equations:

$$\begin{array}{ll} (A) & 3x + 2y - 4z = 3 \\ (B) & 2x + y + 3z = 8 \\ (C) & 5x + 3y + 2z = 14 \end{array}$$

Equation (B) is solved for (y) since its coefficient is one. Substituting this value ($y = 8 - 2x - 3z$) in equations (A) and (C) reduces these equations to:

$$\begin{array}{rl} \begin{array}{r} x + 10z = 13 \\ x + 7z = 10 \\ \hline 3z = 3 \\ z = 1 \\ x = 3 \end{array} & \begin{array}{l} \text{Subtraction} \\ \text{By Substitution} \end{array} \end{array}$$

By substituting these values for (x) and (z) in any one of the given equations the value of the remaining variables is found to be $y = -1$. The solution as outlined above may be extended to include sets of equations involving any number of variables. Furthermore the equations may be of any degree and not necessarily linear as used in the example for the sake of simplicity.

Addition or Subtraction

It is often possible to eliminate one of the variables when solving two equations simultaneously by adding together or subtracting one equation from the other, a previous operation having been performed so that the common variables to be eliminated from each equation have equal coefficients. That one equation can be subtracted from another without destroying the equality of the relationship is apparent from Rule (37). In this instance it is not the same but an equal quantity that is being subtracted from both sides of one of the equations.

SOLUTION OF EQUATIONS

$$(A) \quad 5x^2 + 10y = 85$$

$$(B) \quad 2x^2 - 2y = 10$$

$$5x^2 + 10y = 85$$

$$10x^2 - 10y = 50$$

$$(A) + (B) \quad 15x^2 + 0 = 135$$

$$x^2 = \frac{135}{15} = 9$$

$$x = \sqrt{9} = \pm 3$$

Certain sets of equations in which (x) and (y) occur in the denominators can be solved by the method of addition or subtraction without first eliminating the fractions.

Example.

$$\frac{5}{x} - \frac{6}{y} = -\frac{4}{3}$$

$$\frac{7}{x} + \frac{4}{y} = \frac{13}{3}$$

$$\frac{10}{x} - \frac{12}{y} = -\frac{8}{3}$$

$$\frac{21}{x} + \frac{12}{y} = \frac{39}{3}$$

$$\frac{31}{x} = \frac{31}{3}$$

Rule (41)

$$x = 3$$

$$\frac{35}{x} - \frac{42}{y} = -\frac{28}{3}$$

$$\frac{35}{x} + \frac{20}{y} = \frac{65}{3}$$

$$\frac{62}{y} = \frac{93}{3}$$

$$\frac{2}{3} = \frac{3}{3}$$

$$y = 2$$

In the case of a group of three equations to be solved simultaneously, and in each of the equations appear all three of the variables, the solution consists of combining any one of the equations with each of the other two equations. There will be two new equations thus formed, each containing two unknowns. These two new equations are now solved simultaneously, and the values of the two unknowns determined. The value of the third unknown is found by substituting the values of the two variables into any one of the original equations.

$$\begin{array}{ll}
 \text{(A)} & 2x - y + z = 5 \\
 \text{(B)} & 3x + 2y + 3z = 7 \\
 \text{(C)} & \underline{4x - 3y - 5z = -3} \\
 \text{(A)} & 6x - 3y + 3z = 15 & \text{(B)} & 15x + 10y + 15z = 35 \\
 \text{(B)} & \underline{3x + 2y + 3z = 7} & \text{(C)} & \underline{12x - 9y - 15z = -9} \\
 \text{(D)} & 3x - 5y = 8 & \text{(E)} & \underline{27x + y = 26}
 \end{array}$$

Equations (D) and (E) are now solved simultaneously for (x) and (y), the values of which are substituted in any of the three original equations, and the value of (z) thus determined.

If in a group of three equations to be solved simultaneously, there are but two unknowns appearing in one of the equations, then the other two equations are first solved simultaneously, eliminating the variable not found in the unused equation. The new equation formed will contain but two unknowns, and can be solved simultaneously with the other equation of two variables, and the results used to determine the third variable by a process of substitution in either of the two original equations in which it appears.

$$\begin{array}{ll}
 \text{(A)} & 2x - y + z = 5 \\
 \text{(B)} & 3x + 2y + 3z = 7 \\
 \text{(C)} & \underline{4x - 3y = 7} \\
 \text{(A)} & 6x - 3y + 3z = 15 \\
 \text{(B)} & \underline{3x + 2y + 3z = 7} \\
 \text{(A) } - \text{(B)} & \underline{3x - 5y = 8} \\
 & 3x - 5y = 8 \\
 \text{(C)} & \underline{4x - 3y = 7} \\
 & 12x - 20y = 32 \\
 & \underline{12x - 2y = 21} \\
 & - 11y = 11 \\
 & y = -1 \\
 & x = +1 \quad (\text{by substitution}) \\
 & z = +2 \quad (\text{by substitution})
 \end{array}$$

Errors in solving simultaneous equations by the method of addition and subtraction are the result of:

1. The failure to multiply, or divide, each and every term of the given equation by the same factor in the process of making equal the coefficients of one of the variables common to both equations.
2. Improper addition or subtraction. Both positive and negative terms are of common occurrence, and through carelessness or uncertainty in combining the equations, the corresponding terms are often added together in one case and at the same time, subtracted in another. A thorough understanding of the rules applying to positive (+) and negative (-) numbers is essential.

The following example is correctly solved and may be used as a basis of comparison for similar problems of the same type:

SOLUTION OF EQUATIONS

$$\begin{array}{ll}
 (A) & .866P + 5T = 100 \\
 (B) & .5P - .866T = 0 \\
 (A) (5) & \underline{.433P + .25T = 50} \\
 (B) (.866) & \underline{.433P - .75T = 0} \\
 (A) - (B) & 1.00T = 50 \\
 & T = 50 \\
 (A) (.866) & .75P + 433T = 86.6 \\
 (B) (.5) & \underline{.25P - 433T = 0} \\
 (A) + (B) & 1.00P = 86.6 \\
 & P = 86.6
 \end{array}$$

Comparison

Two equations each involving two unknowns as (x) and (y) may be solved by the method of comparison, which is the application of the following rule:

Solve independently each of the given equations for one of the unknowns—that is, for each equation find (x) in terms of (y) and a constant. Since these two expressions are both equal to (x), they must be equal to each other. This relationship provides an equation involving but one variable (y). Such an equation is readily solved and this numerical value thus found is the value of (y) in either of the two given equations. The value of (x) for both equations is found by substituting (y) in either equation and solving for the remaining unknown variable.

$$\begin{array}{ll}
 (A) & x + y = 15 \quad \text{or } x = 15 - y \\
 (B) & y - x = 1 \quad \text{or } x = y - 1 \\
 \text{then,} & 15 - 1 = y - 1 \\
 & 15 + 1 = y + y = 2y \\
 & y = 8 \\
 & x = 7
 \end{array}$$

Solution of equations by the method of comparison is in reality a solution by the method of Addition and Subtraction previously described. However, it is best to consider it as a separate and distinct solution inasmuch as the procedure, as explained, does not follow the same steps as though it were accomplished by subtracting (B) from (A). The comparison method of solution is especially useful for solving certain types of problems in trigonometry.

Division

The simultaneous solution of two equations by the method of division is an application of Rule (40). In this instance it is not the same but an equal quantity that is used to divide both sides of one of the equations. Before this operation is performed it is advisable to make sure that the value of the expression used as a divisor is not equal to zero, as dividing by zero is never a valid operation.

$$\begin{array}{ll}
 (A) & x^2 + xy = 20 \\
 (B) & x + y = 10 \\
 (A) & x(x + y) = 20 \\
 (B) & (x + y) = 10 \\
 (A) \div (B) & x = 2 \\
 & y = 8 \quad (\text{by substitution})
 \end{array}$$

An additional example employing division is included in Section III—Trigonometry. This method of solution is especially useful for solving certain types of problems in trigonometry.

RATIO, PROPORTION AND VARIATION

Ratio, Percentage, Proportion

Many problems of everyday occurrence are associated with quantities which are dependent upon some other quantity or quantities to determine their absolute numerical values. Such relationships may be either of a mathematical or a non-mathematical nature depending on the characteristics of the factors involved. Relationships of a non-mathematical nature are often expressed by means of a table or by graphs. If a mathematical relation exists among the various factors involved, the problem can be reduced to a mathematical expression which may take the form of either a proportion or a variation. It is to this type of problem that the following notes apply.

The *ratio* of one quantity to another is the result obtained when the first number is divided by the second. Thus the ratio of 5 to 10 is $5/10$ or .5. The ratio of 3 inches to 1 foot (12 inches) is $3/12$ or .25. It is important to know that when obtaining the numerical value of any one ratio it is necessary that both quantities be expressed in the same units, that is, dollars or cents, feet or inches, pounds or ounces, etc., but the result obtained will be an abstract number without units (non-dimentional). Ratios are frequently used in everyday life in stating mechanical advantages, efficiencies, specific gravities, etc.

The value of any particular ratio can always be determined by dividing the first term of the ratio by the second term. When the ratio of a part of any quantity to the whole of that quantity is calculated, the result is always a decimal fraction less than unity. For convenience, such ratios are frequently stated as percentage values, or the number of units representing the fractional part of the quantity when the whole of the quantity is assumed to be 100. The value of a ratio can always be expressed as a percentage by multiplying the absolute value of the given ratio by 100.

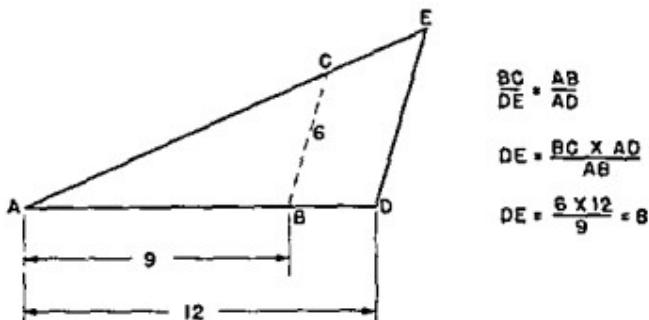
$$\frac{15}{25} = 0.6 = 60\%$$

Whenever the ratio of two quantities is numerically equal to the ratio of two other quantities, an equation can be written to indicate this relationship. This statement of equality between two ratios is termed a *proportion*.

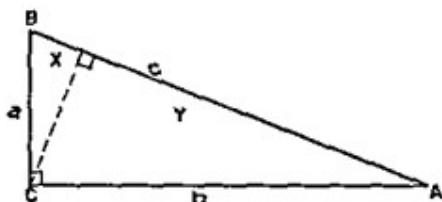
For example, a theorem of geometry (181) states that if the corresponding sides of two triangles are proportional, the triangles are similar. The converse of this theorem is that if two triangles are similar, the corresponding sides are proportional, that is, the same ratio exists among the corresponding sides. By equating any two of the possible three ratios among the corresponding sides of two similar triangles, a proportion can be established.

In the triangle *ADE* shown below, the length of side *DE* is to be determined. Triangle *ABC* is similar to triangle *ADE*.

SOLUTION OF EQUATIONS



In this particular example neither of the terms used in one of the ratios of the proportion is used again in the second ratio. This is usually the case, but in the following example an exception to this is shown. Using the same theorem of geometry as before, two proportions may be employed to advantage in developing a proof of the Pythagorean theorem (Rule 186).



The triangle shown above is a right triangle, defined as a triangle which has one of its interior angles a right angle (90°). The side opposite this right angle is termed the hypotenuse.

A line drawn perpendicular from the hypotenuse to the vertex (corner) of the right angle divides the original right triangle into two smaller right triangles. The smaller right triangle is similar to the original right triangle since each contains one right angle, and angle B is common to both. The third angle of each is also equal since in any triangle the sum of the three interior angles is 180° . Similarly, the second of the smaller right triangles is similar to the original right triangle since each contains one right angle, and angle A is common to both.

Therefore,

$$\frac{x}{a} = \frac{a}{c}$$

also

$$\frac{y}{b} = \frac{b}{c}$$

$$xc = a^2$$

and

$$yc = b^2$$

$$xc + yc = a^2 + b^2 \quad \text{Adding equals to equals, Rule (36)}$$

$$(c)(x+y) = a^2 + b^2$$

but

$$x+y=c$$

therefore

$$c^2 = a^2 + b^2$$

The terms (a) and (b) in the proportions,

$$\cdot \quad \frac{x}{a} = \frac{a}{c} \quad \text{and} \quad \frac{y}{b} = \frac{b}{c}$$

are specifically designated as the *mean* terms of the proportion, and the remaining two terms in each proportion are designated as the extreme terms. These designations are a result of the older and less desirable way of writing the same proportion which would be stated, (x) is to (a) as (a) is to (c) , or algebraically, $x : a :: a : c$. The solution of such a proportion makes use of the fact that the product of the mean terms is equal to the product of the extreme terms. Using the more modern system of denoting ratios as fractions, the first step in the solution is effected by simply finding the cross-product of the terms.

That one of the terms appearing in the first ratio may also be used as one of the terms in the second ratio is apparent by an inspection of the proportions established in the example above. When used in such a manner, the position of the term in the second ratio will be opposite to its position in the first ratio; that is, changed from the numerator to the denominator, or vice versa. Since the same term may be used twice in a single proportion the word "other" as used in the definition of a proportion does not have its usual significance. More specifically, then, a proportion is a statement of equality between *any* two ratios. These ratios may or may not each include a common term.

Another important point which must be understood concerns the dimension of the terms used in writing a proportion. As already stated, each of the two terms of a ratio must be expressed in the same units, but the ratio itself is an abstract number. Therefore in writing a proportion, which is a statement of equality between any two ratios, it is not necessary that the units by which the value of one ratio is determined shall be the same as the units by which the value of the other ratio is determined. An example may be used to demonstrate this point.

If one cubic foot of water weighs 62.4 pounds, what is the weight of 1 gallon (231 cubic inches)?

$$1 \text{ cubic foot} = 1728 \text{ cubic inches.}$$

Let x = weight of 1 gallon (231 cubic inches).

$$\frac{x \text{ lbs.}}{62.4 \text{ lbs.}} = \frac{231 \text{ cu. in.}}{1728 \text{ cu. in.}}$$

$$(1728)(x) = (231)(62.4)$$

$$x = 8.34 \text{ lbs.}$$

If the definitions of ratios and proportions as given are strictly followed there need be no source of error arising from using inconsistent units, inverting one of the ratios, etc. However, the requirements of the definitions can be altered in some instances and the correct result obtained.

In the above example, the solution was obtained by equating the value of one ratio to another. Although such a procedure seems to be the most logical, another method of obtaining the same result is available.

$$\frac{x \text{ lbs.}}{231 \text{ cu. in.}} = \frac{62.4 \text{ lbs.}}{1728 \text{ cu. in.}}$$

$$(1728)(x) = (231)(62.4)$$

$$x = 8.34 \text{ lbs.}$$

Since the cross-products of the two fractions are identical to those obtained by the first method, the solution is correct. However, neither the left nor right side terms of

SOLUTION OF EQUATIONS

the equation constitute a ratio, as they do not satisfy the condition that both numerator and denominator be expressed in the same basis. It should be noticed, however, that the dimensions of the numerator and denominator of one fraction are consistent, or correspond, to the dimensions of the numerator and denominator, respectively, of the second fraction. This condition must always exist.

The equation as written in the second form is not a proportion, since neither of the terms constituting the expression are ratios. There can, therefore, be no equality of ratios. However, such expressions are frequently called proportions, and since the results obtained are valid, the point of argument seems insignificant in such instances. The use of the word ratio is used in another sense not consistent with its mathematical sense. For example, the term strength-weight ratios is commonly used to designate the strength of a substance divided by its weight. Such a fraction contains pounds in the numerator and pounds per square inch in the denominator, and is therefore not a true ratio.

Direct and Inverse Proportions

The expressions *directly proportional* and *inversely proportional* are frequently employed in everyday conversation. When only two variable quantities are being considered, it may be said that they are directly proportional to each other whenever an increase or decrease in one of them produces a proportionate increase or decrease in the other. For example, the height of the column of mercury in a thermometer is directly proportional to the temperature.

Two quantities are inversely proportional to each other if an increase in one of them produces a proportionate decrease in the other, or vice versa. For example, the volume of a fixed weight of gas in a cylinder is inversely proportional to the pressure upon it. The word *inversely* obviously signifies in a reverse order.

The algebraic method of writing a proportion, either direct or inverse, can be demonstrated by using the two examples cited above. For the direct proportion:

$$b \propto T \quad (\propto \text{ means proportional to})$$

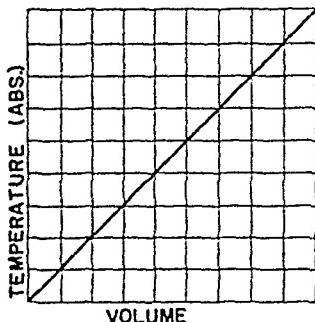
which is read *b* is proportional to *T*, the word *directly* often being omitted when the proportion is of the direct type. For the inverse proportion,

$$V \propto 1/P$$

which is read, *V* is inversely proportional to *P*.

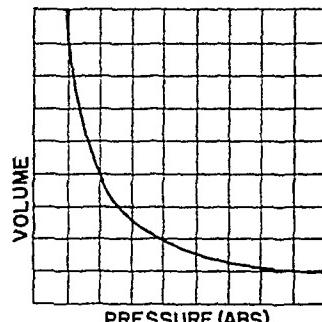
It is at once apparent whether one quantity is directly or inversely proportional to another by observing whether the second quantity appears in the numerator or the denominator of the expression representing the relationship.

Another means of identifying direct proportions and inverse proportions is to observe their graphs. If one quantity is directly proportional to another quantity, a curve representing the relationship will plot as a straight line. If the proportion is an inverse relationship, the curve will plot as a hyperbola. The gas laws known as Charles' and Boyle's laws clearly demonstrate this point.



Charles' Law

$$\frac{V_1}{V_2} = \frac{T_1}{T_2}$$



Boyle's Law

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}$$

Variation

The ideas of *variation* are much the same as in ratio and proportion but in many ways are far more convenient and practical. In mathematical expressions both constants and variables are encountered. As its name suggests, a constant is a number whose magnitude does not change, while a variable quantity may have an unlimited number of values. Constants are represented algebraically by the first letters of the alphabet, (*a*), (*b*), (*c*), etc., or more often by *K*. Variables are usually represented by (*x*), (*y*), (*z*), etc. Where one quantity (*x*) varies directly as another quantity (*y*), the relationship can be written:

$x \propto y$ means proportional to, or varies as.

This is nothing more than a statement of proportionality which may be rewritten as an equation by introduction of a *constant variation*.

$$x = Ky$$

or $K = x/y$

The numerical value of *K* may be found whenever two instantaneous values of (*x*) and (*y*) are obtained, and, once determined, its value remains unchanged throughout the relationship. Fundamentally, variation means nothing more than finding the constant which connects two or more related variables.

The statements, varies directly and varies inversely, have the same significance as direct proportion and inverse proportion. If one quantity varies *directly* as another, an increase or decrease in either one of them will produce a proportionate increase or decrease in the other. If one quantity varies *inversely* as another, an increase in either one of them will produce a proportionate decrease in the other, and similarly, if one decreases the other one increases. The term *inversely* has the same significance as indirectly, both terms signifying—in a reverse order. Whether a quantity varies directly or inversely as another is indicated by the position of the second quantity in the equation. In the relation $P = Kb$, the value of *P* will vary directly as (*b*), while in equation $P = K/L$, *P* will vary inversely as *L*.

It should be observed that if one quantity varies directly as another, the quotient of one quantity divided by the other will always be a constant.

$$P = KL$$

$$\frac{P}{L} = K$$

Also, if one quantity varies inversely as another, the product of the two quantities will always remain constant.

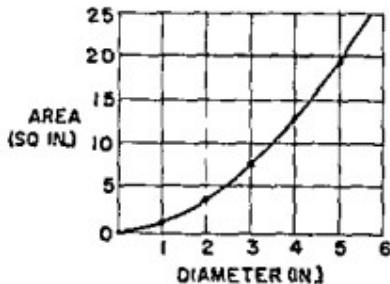
$$P = K/L$$

or $PL = K$

Direct and inverse variations have the same characteristics when plotted as direct and inverse proportions. Because a direct variation always appears as a straight line, such functions are termed straight line or linear variations.

It is often apparent that one quantity varies, either directly or inversely, at some rate other than the first power of the second variable. For instance, the area of a circle varies as the square of either the radius or the diameter.

$$\text{Area} = 3.1416R^2 = 3.1416 \left(\frac{D}{2}\right)^2 = 7854D^2$$



In such cases, the equation involving the two variables will not plot as a straight line, but as some type of curve. For example, the equation representing the variation of the area of a circle with the diameter appears as a parabola as shown above. If the exponent of the second variable had been greater than 2, the curve would be steeper (greater slope) and if less than 2, the curve would be flatter (less slope). If the exponential term appears in the denominator, as in the case of an inverse variation, the equation will plot as a hyperbola. (See curve B, Page 92).

Joint Variation

In many cases, the magnitude of one quantity is dependent not on one but on several other quantities. Furthermore, the quantity may vary directly as one or more of the quantities, and at the same time vary inversely with respect to the others. The strength of a simple beam can be used to demonstrate this point. Other things being equal, the strength of the beam varies (1) directly as its breadth, (2) directly as the square of its depth, and (3) inversely as its length.

$$S = K_1 b$$

$$S = K_2 (d)^2$$

$$S = \frac{K_3}{L}$$

These three equations may be combined to form a single equation to represent the strength of the beam if the breadth, depth and length are all changed:

$$S = \frac{Kbd^2}{L}$$

This is mathematically correct according to the law of joint variation which states that if a quantity varies directly as two or more quantities, it varies directly as their product; and if a number varies directly as one quantity and inversely as another, it then varies as the quotient of the first divided by the second.

Section II
GEOMETRY

INTRODUCTION

Geometry is a study of the measurement of lines, angles, surfaces, and solids, with their various relations. In plane geometry only figures which lie in one plane are considered. The term plane will not be repeated in the paragraphs which follow, but should be understood to apply in all cases, as only plane geometry is being discussed.

Many of the facts used in geometry are common knowledge which do not require any mathematical proof. These fundamental statements are called axioms and upon them the various propositions of geometry are established. Axioms may be divided into two groups, which are:

General axioms, or axioms which apply to other kinds of quantities as well as to geometric magnitudes, for instance, to numbers, forces, masses.

Geometric axioms, or axioms which apply to geometric magnitude alone.

The general axioms may be stated as follows:

- (48). Things which are equal to the same thing, or to equal things, are equal to each other.
- (49). Any quantity may be substituted for its equal in any process.
- (50). If equals are added to equals, the sums are equal.
- (51). If equals are subtracted from equals, the remainders are equal.
- (52). If equals are multiplied by equals, the products are equal.
- (53). If equals are divided by equals, the quotients are equal.
- (54). Like powers or like roots of equals are equal.
- (55). The whole is greater than any of its parts.
- (56). The whole is equal to the sum of its parts.

The geometric axioms may be stated as follows:

- (57). Through or connecting two given points, only one straight line can be drawn.
- (58). A geometric figure may be freely moved in space without any change in form or size.
- (59). Through a given point outside a given straight line, one straight line, and only one, can be drawn parallel to the given line.
- (60). Geometric figures which can be made to coincide are congruent, that is, they are equal figures.

A geometric postulate is a construction of a geometric figure which, without proof, is admitted as possible.

The geometric postulates may be stated as follows:

- (61). Through or connecting any two points, a straight line may be drawn.

- (62) A straight line may be extended indefinitely, or may be limited at any point.
- (63) A circle may be described about any given point as a center, and with any given radius

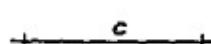
Beside the postulates which are used in the actual construction of figures, there are certain other postulates which are used only in the processes of reasoning. As an example, a given angle may be regarded as divided into any convenient number of equal parts. Whether it is possible to actually divide this angle on paper by use of the ruler and compass, depends upon practical limitations.

In the paragraphs below, the various terms used in geometry are defined and explained. This is followed by a listing of the most important of the geometric propositions.

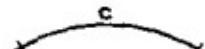
DEFINITIONS

There are many familiar terms which, in geometry, are used with such exactness as to require a precise definition. The following terms defined below should be clearly understood and should be consulted from time to time.

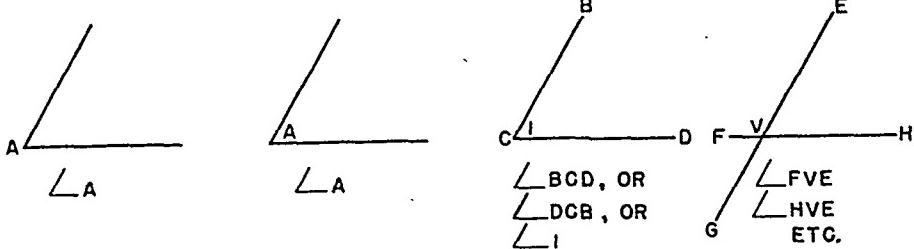
- (64). A *straight line* is the path of a point moving always in one direction. Lines are considered to have length only, not breadth or thickness. A straight line is considered as unlimited in its length. The word *line-segment* refers to a line of definite length, or to a part of an unlimited straight line between two of its points. A line-segment is identified by two letters, one placed at each end, or by a single letter placed somewhere along its length. Arbitrarily chosen capital letters are usually used at the ends of the line, while a single small letter if used is placed along the length of the line.



- (65) A *curved line* is the path of a point moving in a continually changing direction.

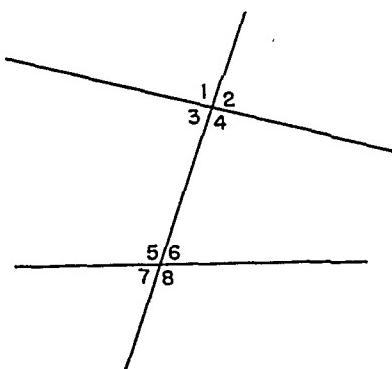


- (66). An *angle* is the figure formed by two straight lines drawn from the same point. The point is called the *vertex* of the angle, and the bounding straight lines are called the *sides* (or *legs*) of the angle. An angle is usually identified by an arbitrarily chosen capital letter placed at the vertex, or by capital letters at the vertex and at the ends of the sides. Numerals may be similarly employed. The symbol \angle is used for the word, angle; $\&$, for angles.



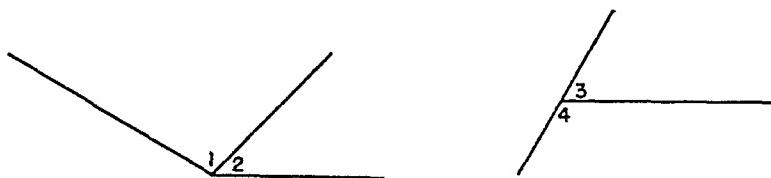
Angles, as used in geometry, are measured in degrees and their subdivisions as explained on page 83.

If two straight lines are cut by a third, called a transversal, the angles are named as follows:



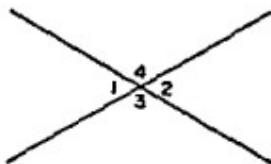
- ∠ 1, 2, 7, and 8 are *exterior* angles.
- ∠ 3, 4, 5, and 6 are *interior* angles.
- ∠ 1 and 8, and 2 and 7, are pairs of *alternate-exterior* angles.
- ∠ 3 and 6, and 4 and 5, are pairs of *alternate-interior* angles.
- ∠ 1 and 5, 2 and 6, 5 and 7, and 4 and 8, are pairs of *exterior-interior* angles, often called *corresponding angles*.

- (67). Two angles which have the same vertex and a common side between them are called adjacent angles. Thus, ∠ 1 and 2 are adjacent angles, also ∠ 3 and 4 are adjacent.

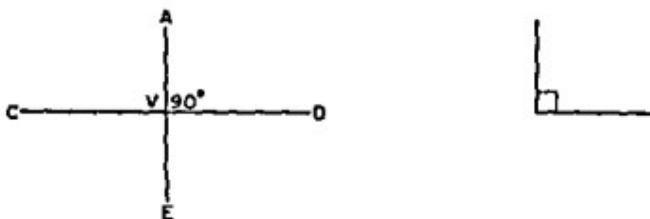


- (68). Two angles which have the same vertex and the sides of one are the prolongations of the sides of the other are called *vertical angles*. Thus ∠ 1 and 2 are vertical angles, also ∠ 3 and 4 are vertical angles.

SOLUTION OF EQUATIONS

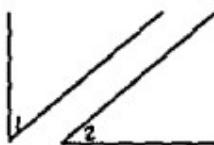
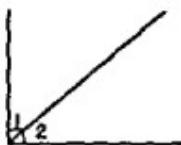


- (69) If two adjacent angles formed by the intersection of two straight lines are equal, each angle is a *right angle* or 90° . Right angles are indicated hereafter by a small square at the vertex of the angle.

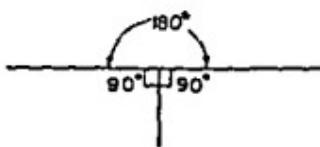


Two straight lines which intersect to form right angles are said to be *perpendicular* to each other, or *normal* to each other. The point of intersection of the perpendicular with the given line is called the *foot of the perpendicular*.

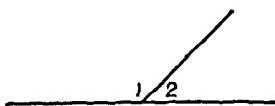
- (70) An *acute angle* is an angle smaller than a right angle, that is, less than 90° .
- (71) An *obtuse angle* is an angle larger than 90° , but not greater than 180° .
- (72) Two angles whose sum is one right angle, or 90° , are called *complementary angles*, and either one is said to be the complement of the other. Thus, $\angle 1$ and 2 are complementary angles. Angles need not be adjacent to be complementary.



- (73) Two sides of an angle which extend in opposite directions from the vertex form a *straight angle*. Such an angle contains two right angles, or 180° .



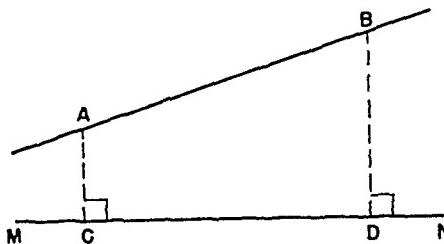
- (74) Two angles whose sum is one straight angle, or 180° , are called *supplementary angles*, and either one is said to be the supplement of the other. Thus, $\angle 1$ and 2 are supplementary angles. The angles are not necessarily adjacent.



- (75). A *plane surface*, or *plane*, is a surface on which any two points can be connected by a straight line lying wholly in the surface.
- (76). Two straight lines lying in the same plane which do not meet no matter how far extended are parallel.



- (77). The *projection* of a line segment upon a second line in the same plane is the segment of the second line included between the perpendiculars drawn from it to the extremities of the given line segment. Thus, *CD* is the projection on line *MN* of the line segment *AB*.



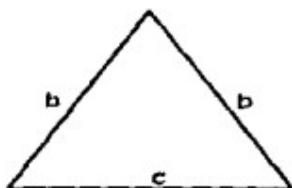
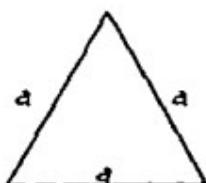
- (78). A *polygon* is a plane surface bounded by three or more straight lines. Any polygon contains the same number of angles as it has sides. A *regular polygon* is one that is both equilateral and equiangular, that is, the sides are of equal length and the angles are of equal magnitude. A *diagonal* of a polygon is a straight line jointing any two vertices not adjacent to each other. The *perimeter* of a polygon is the sum of the lengths of its sides.

- (79). A *triangle* is a polygon of three sides and consequently three angles. The side connecting any two given angles and common to both angles is called the *included side*. The angle formed by any two given sides of a triangle is called the *included angle*. Triangles which are of exactly the same size and shape are said to be *congruent*, since they can be made to coincide.

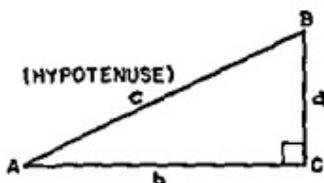
Triangles which are exactly the same shape, but of different size, are called *similar triangles*. The corresponding angles of similar triangles are equal, and the corresponding sides are proportional.

Any triangle that has three equal sides and three equal angles is designated as an *equilateral triangle*. An *isosceles triangle* is one having only two of its sides of equal length.

SOLUTION OF EQUATIONS



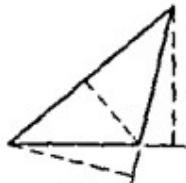
- (80). A *right-angled triangle*, or *right triangle*, is one which has one of its interior angles equal to the right angle, or 90° . The side of the triangle opposite the right angle is the *hypotenuse*. The length of this side is greater than the length of either one of the other two sides, but is always less than the sum of their lengths. The two sides of a right triangle other than the hypotenuse are called *legs*.



- (81) An *oblique triangle* is one which does not have one of its interior angles a right angle measuring 90° . Such triangles may or may not contain one angle greater than 90° .



- (82) The *altitude* of a triangle is the perpendicular distance from any side (extended, if necessary) to the vertex of the angle opposite that side. A line drawn from the vertex of an angle to the mid-point of the opposite side is called the *median* line. A line drawn through the vertex of an angle and dividing the angle into two equal parts is called the *bisector* of the angle.



Altitudes



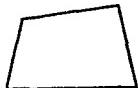
Medians



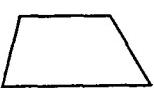
Bisectors

- (83). A *quadrilateral* is a polygon of four sides. If two of the sides are parallel, the figure is a *trapezoid*. The parallel sides are called *bases*, and the

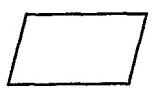
altitude is the perpendicular distance between the two bases. A *parallelogram* is a quadrilateral having both pairs of opposite sides parallel. A diagonal divides a parallelogram into two equal triangles. A *rectangle* is a parallelogram whose angles are all right angles. A parallelogram in which the sides are all of the same length is called a *rhombus*.



Quadrilateral



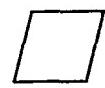
Trapezoid



Parallelogram



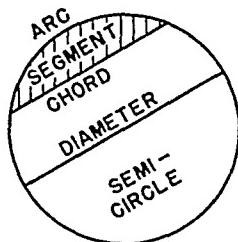
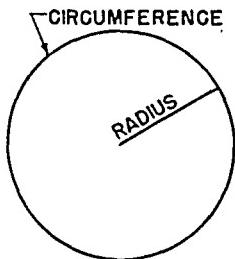
Rectangle



Rhombus

(84). Other polygons frequently encountered in geometry have five, six, and eight sides. These are known as *pentagons*, *hexagons*, and *octagons*, respectively.

(85). A *circle* is a plane figure bounded by a closed curved line called the *circumference*, every part of which is the same distance from a fixed point within, called the *center*. The distance from the center to the circumference is the *radius*. Two circles drawn from the same center, but with different radii, are said to be *concentric*.

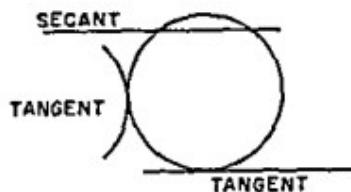


Any part of the circumference is called an *arc*, and a straight line joining the two ends of the arc is called a *chord*. The chord is said to subtend its arc. The area of the circle bounded by an arc and its chord is referred to as a *segment* of the circle.

The circumference of a circle can be divided into any number of arcs. Whenever only two arcs of unequal length are found, the longer arc is called the *major arc*, and the shorter arc is called the *minor arc*.

The chord which passes through the center of the circle is the *diameter*. The diameter, therefore, divides the circle into two equal parts called *semicircles*. The length of the diameter is twice the length of the radius. The length of the circumference of a circle divided by the diameter is an irrational number designed by π (Pi), the approximate value of which is 3.1416. The length of the circumference of a circle in terms of the radius or diameter is therefore:

$$\text{Circumference} = \pi D = 2\pi R.$$



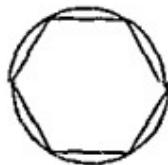
A *secant* to a circle is any straight line intersecting the circle. A *tangent* to a circle is a line which touches the circumference at only one point. A tangent line may be either a straight line or a curved line.

- (86) A *central angle* is an angle having its vertex at the center of the circle and radii of the circle for its sides. The angle is said to *intercept* the arc included between its sides. Thus, $\angle ABC$ intercepts \widehat{AB} (read, arc AB). The area of a circle bounded by the two radii of a central angle and the intercepted arc is called a *sector* of a circle.

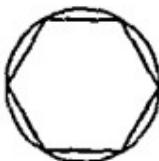


An *inscribed angle* is an angle within the circle whose vertex lies on the circumference and whose sides are chords of the circle. An angle is said to be *inscribed in a segment* if the vertex is on the circumference and the sides pass through the ends of the arc of the segment.

- (87) A *regular circumscribed polygon* is a regular polygon having all of its sides tangent to a circle. The *apothem* of a regular polygon is the radius of its *inscribed circle*.



- (88). A *regular inscribed polygon* is a regular polygon having all of its vertices on the circumference of a circle.

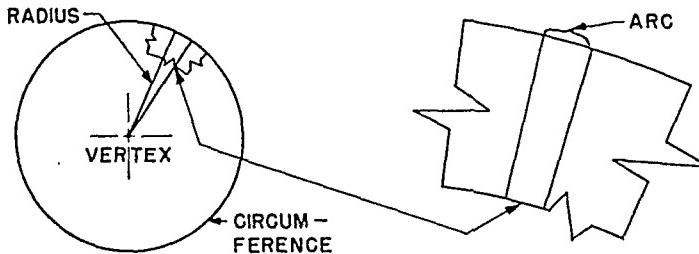


MEASUREMENT OF ANGLES

There are two methods of measuring angles in common use, the sexagesimal method and the circular or natural method. The natural measurement of angles is not essential to the study of geometry, but is here included for completeness.

Sexagesimal Measurement

Sexagesimal measurement of angles is the method most familiar to everyone. It is usually thought of as the Degrees-Minutes-Seconds System as these are the units in which the angles are measured. These units are derived as follows:



A circle is drawn by revolving a line, called a radius, about a fixed point called the vertex or pole. Radial lines are drawn from the vertex so that the circumference is divided into 360 equal arcs. The length of one of these arcs will depend upon the length of the radius; but the angle at the center subtended by one of these divisions will be independent of the radius since it is $1/360$ of 360 degrees, a complete circle, or 1 *degree* ($^{\circ}$). This unit is divided into sixty parts called *minutes* ('), each of which is subdivided into sixty parts called *seconds* (''). Any angle such as 30 degrees 20 minutes and 15 seconds may be written $30^{\circ} 20' 15''$. In many cases it is preferable to define the angle in units of degrees and minutes only. Thus, $30^{\circ} 20' 15''$ is also written $30^{\circ} 20.25'$ since $15 \text{ seconds} = .25 \text{ minutes}$. This latter form is the more convenient for mathematical computations as tables of natural trigonometric functions are usually graduated by degrees and minutes only, with no mention of any such term as seconds. Furthermore, retaining the fraction of a minute as a decimal simplifies to some extent operations as described below.

Regardless of the method of indicating fractional parts of a minute, either as decimal parts of a minute, or as seconds, the operation of adding, subtracting, multiplying or dividing angles measured in the sexagesimal system is somewhat complicated inasmuch as the degrees-minutes-seconds units do not comprise a decimal system. These operations are explained by examples as follows:

When adding angles, place the degrees, minutes, and seconds under each other and add the individual columns. If the operation produces a sum of minutes or seconds greater than sixty, take sixty or a multiple of sixty from that column and for each add one to the column on its left, in order that the answer may be written in its simplest form. Thus to add:

$$\begin{array}{r}
 25^{\circ} 30' 35'' \\
 35^{\circ} 40' 45'' \\
 \hline
 60^{\circ} 70' 80'' = 60^{\circ} 71' 20'' = 61^{\circ} 11' 20''
 \end{array}$$

When subtracting angles, place the degrees, minutes, and seconds under each other and subtract the individual columns. If there are not sufficient minutes or seconds in

SOLUTION OF EQUATIONS

the upper number of the column, take one from the column directly to the left of it and add sixty to the insufficient number, as in the following example.

$$\begin{array}{r} 25^{\circ} 45' 15'' = 25^{\circ} 44' 75'' \\ 15^{\circ} 25' 35'' \quad 15^{\circ} 25' 35'' \\ \hline 10^{\circ} 19' 40'' \end{array}$$

When multiplying angles, multiply each column by the required number, and if the seconds or minutes columns are over sixty, they are reduced the same as in addition. The following example indicates this method

$$\begin{array}{r} 10^{\circ} 30' 40'' \\ \times 2 \\ \hline 20^{\circ} 60' 80'' = 20^{\circ} 61' 20'' = 21^{\circ} 1' 20'' \end{array}$$

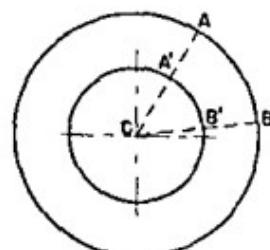
To divide an angle into a given number of parts, divide each column by the number, starting with the degree column. The remainder of the dividend is converted into minutes and added to the minutes column, which is then divided by the number. This remainder is then converted into seconds, added to the seconds column, and divided by the number. The procedure is shown in the following example, in which an angle $243^{\circ} 50' 30''$ is divided by 4

$$\begin{array}{r} 60^{\circ} & 57' & 37.5'' \\ \hline 4/243 & & \\ 240 & 3 \times 60 = 180 & 30 \\ \hline 3 & 4/230 & 2 \times 60 = 120 \\ & 228 & \hline & 4/150 & \\ & 2 & \end{array}$$

It is not always necessary to perform the operation described above. These operations are dependent upon the magnitude of the units involved. However, to minimize the effort which may be required in such instances, it is advisable to maintain the angles involved in the simplest dimensions possible, that is, in degrees, minutes, and fractions of minutes expressed decimalily

Natural Measurement

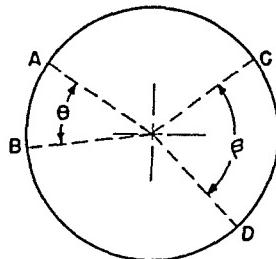
The circular or natural measurement of angles is frequently employed in mechanics. Instead of dealing with degrees-minutes-seconds, a new unit termed *radian* is employed. Do not confuse a radian with a radius. A radian is not an arc or a line, but an angle.



By geometric reasoning, and from common observation as well, it is known that in any two concentric circles, the arcs intercepted by any angle having its vertex at the center of the circles bear the same relationships to each other as do the radii of the circles. Therefore if ACB is any central angle, then,

$$\frac{\text{arc } AB}{AC} = \frac{\text{arc } A'B'}{A'C} = \text{constant} = k$$

from which it is apparent that the length of the intercepted arc divided by the radius is a number that is always the same for the *same* angle no matter what the radius may be. This is true because the length of the arc at any given angle varies directly as the radius.



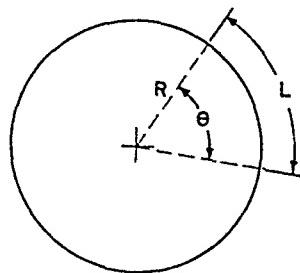
β is the Greek letter Beta.
 θ is the Greek letter Theta.

It is also known that in a circle any two central angles are to each other as their intercepted arcs, and consequently as the quotients of their intercepted arcs divided by the radius (since the radius is the same in both cases).

$$\frac{\theta}{\beta} = \frac{\text{arc } AB}{\text{arc } CD} = \frac{\text{arc } AB/R}{\text{arc } CD/R}$$

$$\text{or } \theta = \frac{\text{arc } AB}{R} \quad \beta = \frac{\text{arc } CD}{R}$$

These two quotients as described lead to the definition of circular measurement: The circular measure (in radians) of an angle in a circle is the quotient obtained by dividing the length of its intercepted arc by the radius of the circle.



Thus if θ is the circular measure of the angle (in radians), L its intercepted arc, and R the radius of the circle,

$$\theta = \frac{L}{R}$$

It is apparent that any angle whose measure is one radian will intercept an arc equal in length to the radius, or conversely, an arc equal in length to the radius sub-

tends a central angle of one radian. The magnitude, in radians, of any other angle can be found by dividing the intercepted arc by the radius. Conversely, the length of the intercepted arc is:

$$L = R\theta$$

This relationship is somewhat simpler than if the angle were expressed in degrees. If the radius of the circle is assumed to be unity,

$$\theta = \frac{L}{R} = \frac{L}{1} = L$$

And it may, therefore, be said that the circular measure of an angle is represented by the length of the intercepted arc in a circle whose radius is unity.

Conversion of Units

Since the circumference of a circle is equal to $2\pi R$, the central angle (in radians) will be

$$\frac{2\pi R}{R} = 2\pi \text{ radians}$$

In sexagesimal measure a complete circle consists of 360° .

The relation between the two systems of measurement is

$$2\pi \text{ Radians} = 360^\circ$$

$$\pi \text{ Radians} = 180^\circ$$

$$\text{or} \quad 1 \text{ Radian} = \frac{180^\circ}{3.1416} = 57.3^\circ \quad (\text{Approximately})$$

$$\text{or} \quad 1^\circ = \frac{3.1416}{180} = .01745 \text{ Radians.}$$

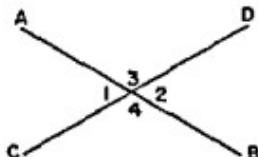
PROPOSITIONS—GEOMETRIC THEOREMS AND PROBLEMS

The term proposition as used in geometry is a general term which includes both geometric theorems and geometric problems. A geometric theorem is a statement of truth concerning geometric objects which requires demonstration. A geometric problem is a statement of the construction of a geometric figure, which is required to be made. There are also statements called corollaries which are evident truths as a direct consequence of either an axiom or a proposition.

The most important of the geometric propositions and corollaries are listed in the following pages. The proof of these statements is not included, but any geometry book will give them completely. It is the propositions themselves, and not their proofs, that are of practical value.

Rectilinear Figures

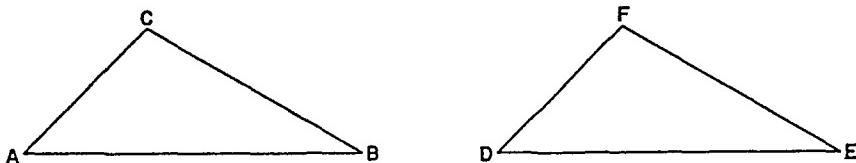
(89). Theorem. If two lines intersect, the vertical angles are equal.



Given the straight lines AB and CD intersecting to form two pairs of vertical angles.

Angle 1 equals angle 2, and angle 3 equals angle 4.

- (90). Theorem: Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.

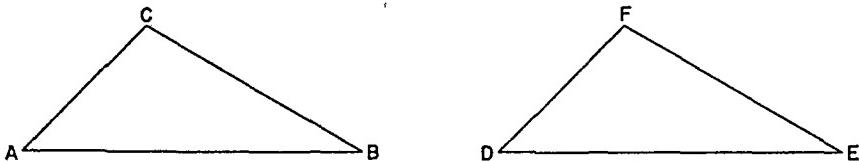


Given the triangles ABC and DEF , with AC equal to DF , AB equal to DE , and angle A equal to angle D .

Triangle ABC is congruent to triangle DEF .

Corollary: Two right triangles are congruent if the legs of the first are equal respectively to the legs of the second.

- (91). Theorem: Two triangles are congruent if two angles and the included side of one are equal respectively, to two angles and the included side of the other.

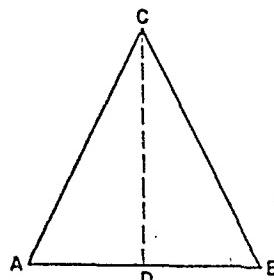


Given the triangles ABC and DEF , with angle A equal to angle D , angle B equal to angle E , and AB equal to DE .

Triangle ABC is congruent to triangle DEF .

Corollary: Two right triangles are congruent if a leg and adjacent acute angle of the first are equal respectively to a leg and adjacent acute angle of the second.

- (92). Theorem: If two sides of a triangle are equal, the angles opposite these sides are equal, that is, the base angles of an isosceles triangle are equal.

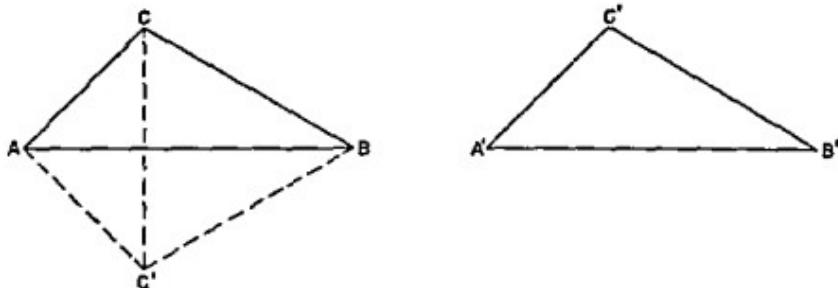


Given the isosceles triangle ABC in which AC is equal to BC . Angle A is equal to angle B .

SOLUTION OF EQUATIONS

Corollary: The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.
 Corollary: An equilateral triangle is also equiangular.

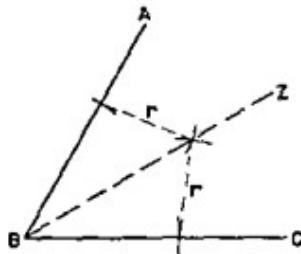
- (93). Theorem. Two triangles are congruent if three sides of one are equal respectively to three sides of the other.



Given the triangles ABC and $A'B'C'$, with AB equal to $A'B'$, AC equal to $A'C'$, and CB equal to $C'B'$.

Triangle ABC is congruent to triangle $A'B'C'$

- (94) Problem To bisect a given angle



Given angle ABC .

Line BZ bisects angle ABC .

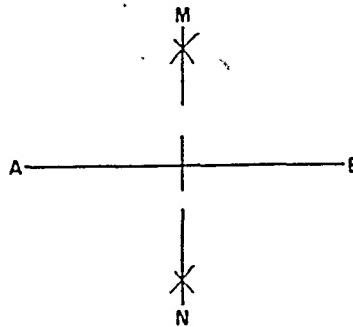
- (95) Problem: At a point on a line construct an angle equal to a given angle.



Given line AB and the point P in AB ; also, the angle D .

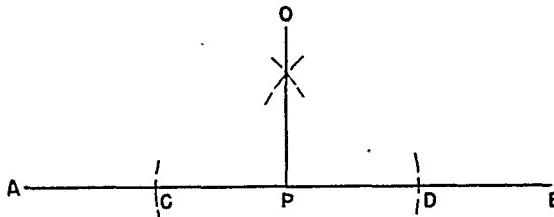
Angle P is equal to angle D .

- (96). Problem: To bisect a given line segment; or, to construct the perpendicular bisector of a given line segment.



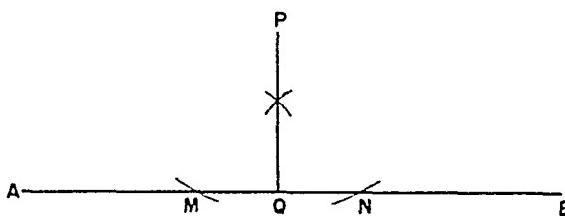
Given the line segment AB .
 MN bisects line segment AB .
 to the line.

- (97). Problem: At a given point in a straight line construct a perpendicular
 Given the line AB and the point P in the line.



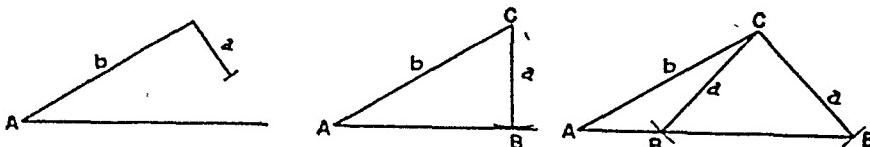
OP is perpendicular to AB at point P .

- (98). Problem: To draw a perpendicular to a given line from a given external point.

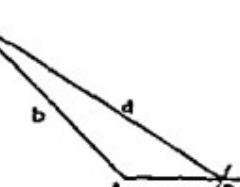
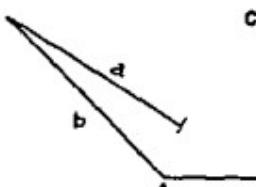
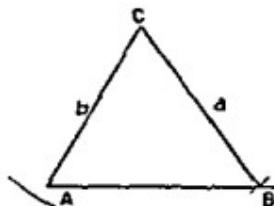


Given the line AB and the point P outside AB .
 PQ is perpendicular to AB from point P .

- (99). Problem: To construct a triangle, having given two sides and the angle opposite one of them.



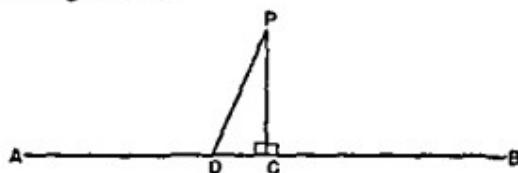
SOLUTION OF EQUATIONS



Given (a) and (b) sides of a triangle, and $\angle A$ the angle opposite side (a). Angle A may be either acute or obtuse. The various possible and impossible triangles are shown.

Triangles ABC are formed

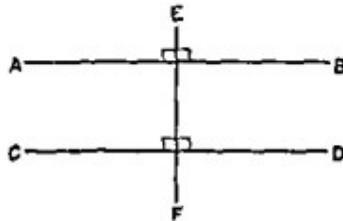
- (100) Theorem Only one perpendicular can be drawn from a given external point to a given line



Given the line AB and the external point P . PC is perpendicular to AB , and PD is any other line from P to AB .

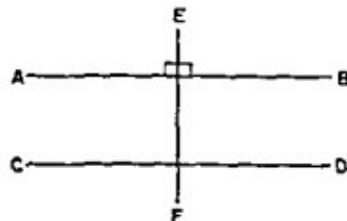
PD is not perpendicular to AB

- (101). Theorem Two lines in the same plane perpendicular to the same line are parallel.



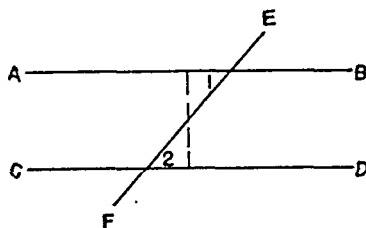
Given two lines AB and CD , in the same plane, both perpendicular to EF .
 AB is parallel to CD .

- (102). Theorem If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.



Given AB parallel to CD , and EF perpendicular to AB .
 EF is perpendicular to CD .

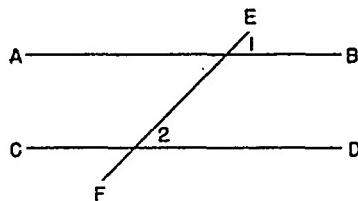
- (103). Theorem: If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Given two parallel lines AB and CD cut by the transversal EF to form alternate interior angles 1 and 2.

Angle 1 is equal to angle 2.

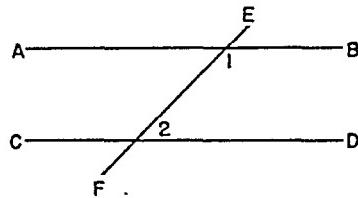
- (104). Theorem: If two parallel lines are cut by a transversal, the corresponding angles are equal.



Given two parallel lines AB and CD cut by the transversal EF to form corresponding angles 1 and 2.

Angle 1 is equal to angle 2.

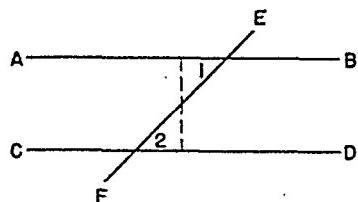
- (105). Theorem: If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.



Given two parallel lines AB and CD cut by the transversal EF to form the interior angles on the same side of the transversal 1 and 2.

Angle 2 is supplementary to angle 1.

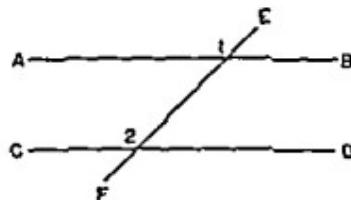
- (106). Theorem: If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.



SOLUTION OF EQUATIONS

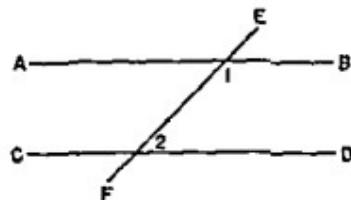
Given two lines AB and CD cut by the transversal EF so that the alternate interior angles 1 and 2 are equal.
 AB is parallel to CD .

- (107). Theorem: If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.



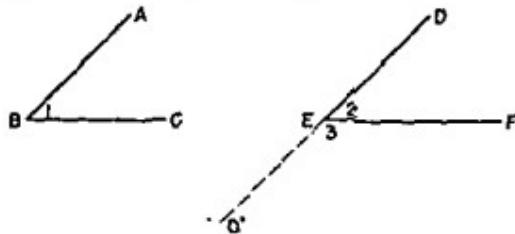
Given two lines AB and CD cut by the transversal EF so that the corresponding angles 1 and 2 are equal.
 AB is parallel to CD .

- (108). Theorem: If two lines are cut by a transversal so that two interior angles on the same side of the transversal are supplementary, the lines are parallel



Given two lines AB and CD cut by the transversal EF so that the two interior angles on the same side of the transversal 1 and 2 are supplementary
 AB is parallel to CD .

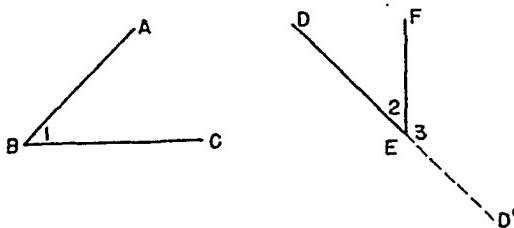
- (109). Theorem: If the sides of one angle are parallel to the sides of another, the angles are either equal or supplementary.



Given the angles ABC and DEF , with DED' parallel to AB , and EF parallel to BC .

Angle 2 is equal to angle 1, and angle 3 is supplementary to angle 1

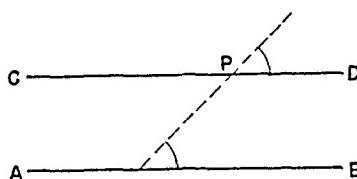
- (110). Theorem: If the sides of one angle are perpendicular to the sides of another, the angles are either equal or supplementary.



Given the angles ABC and DEF , with DED' perpendicular to AB , and EF perpendicular to BC .

Angle 2 is equal to angle 1, and angle 3 is supplementary to angle 1.

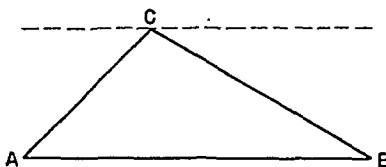
- (111). Problem: Through a given point, to draw a straight line parallel to a given straight line.



Given the point P outside the line AB .

CPD is parallel to AB .

- (112). Theorem: The sum of the three angles of any triangle is equal to 180° .



Given the triangle ABC .

Angle A plus angle B plus angle C is equal to 180° .

Corollary: The sum of any two angles of a triangle is less than 180° .

Corollary: If one angle of a triangle is a right angle or an obtuse angle, each of the other two angles of the triangle must be acute.

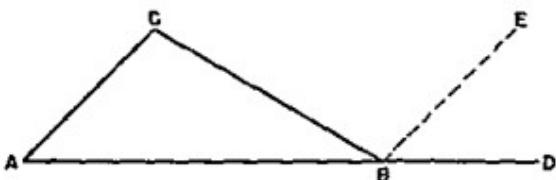
Corollary: In a right triangle, the sum of the two acute angles equals 90° . Each angle of an equilateral triangle contains 60° .

Corollary: If two angles of one triangle are respectively equal to two angles of another triangle, the third angle of the first triangle equals the third angle of the second.

Corollary: If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the triangles are equal.

- (113). Theorem: An exterior angle of a triangle is equal to the sum of the opposite interior angles, and is therefore greater than either of them.

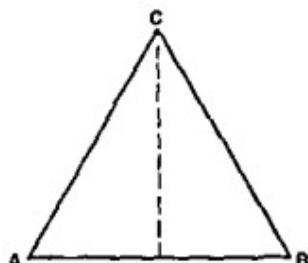
SOLUTION OF EQUATIONS



Given the triangle ABC with AB extended to form the exterior angle CBD .

Angle CBD is equal to angle A plus angle C .

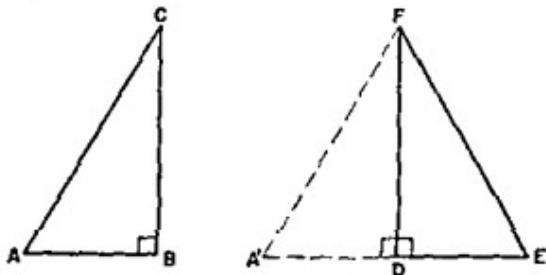
- (114) Theorem. If two angles of a triangle are equal, the sides opposite these angles are equal.



Given the triangle ABC in which angle A is equal to angle B .
AC is equal to BC.

Corollary. An equiangular triangle is also equilateral.

- (115) Theorem. If two right triangles have the hypotenuse and a leg of one respectively equal to the hypotenuse and a leg of the other, the triangles are congruent

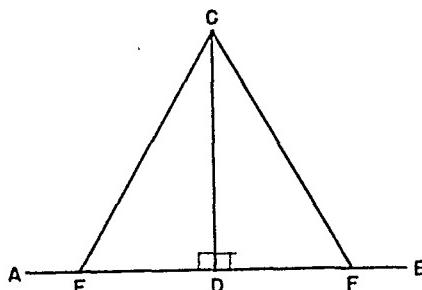


Given hypotenuse AC equal to hypotenuse EF , and leg CB equal to leg FD .

Right triangles ABC and DEF are congruent.

Corollary: Two right triangles are congruent if the hypotenuse and an acute angle of the first are equal respectively to the hypotenuse and an acute angle of the second

- (116). Theorem: Two straight lines drawn from a point in a perpendicular to a given line which meet the line at equal distances from the foot of the perpendicular are equal and make equal angles with the perpendicular

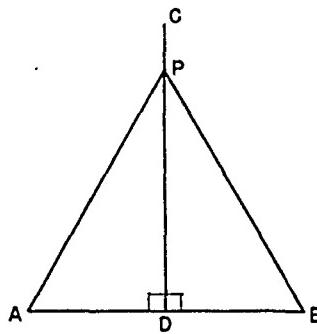


Given: CD perpendicular to line AB , and oblique lines CE and CF cutting off equal segments ED and DF .

CE is equal to CF , and angle ECD is equal to angle FCD .

Corollary: If equal lines are drawn from a point in a perpendicular to a given line, they cut off equal segments on that line from the foot of the perpendicular.

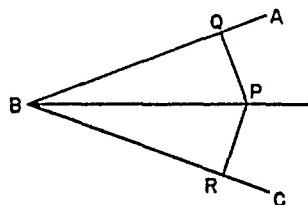
- (117). Theorem: Every point in the perpendicular bisector of a line segment is equidistant from the extremities of the line segment.



Given CD the perpendicular bisector of line AB , and P any point in CD . PA is equal to PB .

Corollary: Conversely, every point equidistant from the extremities of a line lies in the perpendicular bisector of the line.

- (118). Theorem: Every point in the bisector of an angle is equidistant from the sides of the angle.

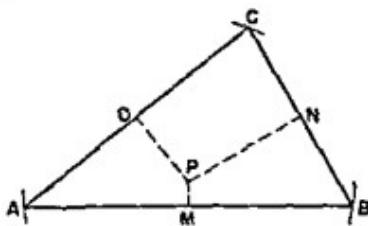


Given PB the bisector of the angle ABC , P any point in PB , PQ perpendicular to BA , and PR perpendicular to BC .

PQ is equal to PR .

Corollary: Conversely, every point equidistant from the sides of an angle lies in the bisector of the angle.

- (119) Theorem: The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the vertices of the triangle.



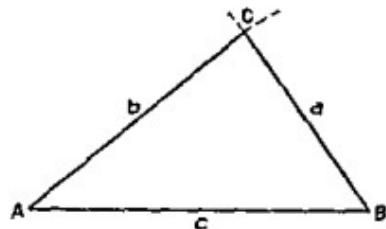
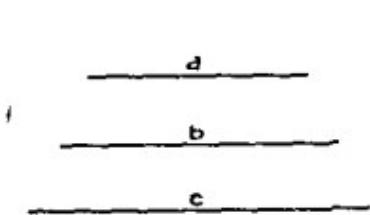
Given the triangle ABC with MP , NP , and OP the perpendicular bisectors of the sides

MP , NP , and OP meet at point P which is equidistant from the vertices of triangle ABC .

Corollary The bisectors of the angles of a triangle meet in a point which is equidistant from the three sides of the triangle

Corollary The three altitudes of a triangle meet in a point

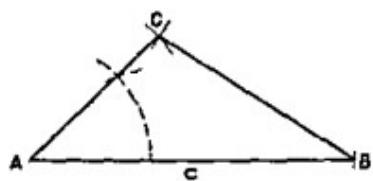
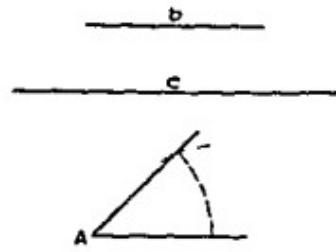
- (120) Problem To construct a triangle having its three sides respectively equal to three given straight lines.



Given the lines (a) , (b) , and (c)

In the constructed triangle ABC , AB is equal to line (c) , BC is equal to line (a) , and AC is equal to line (b) .

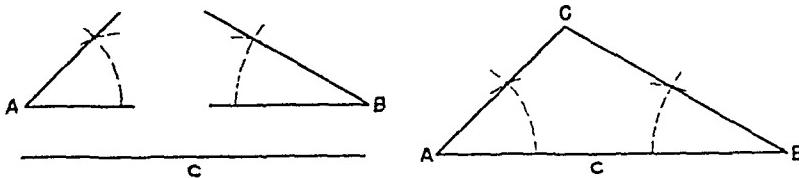
- (121) Problem To construct a triangle having two sides and the included angle respectively equal to two given lines and a given angle.



Given the lines (b) and (c) , and the angle A .

In the constructed triangle ABC , AB is equal to line (c) , AC is equal to line (b) , and angle A is equal to the given angle A .

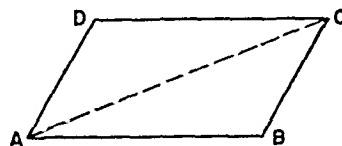
- (122). Problem: To construct a triangle having two angles and the included side respectively equal to two given angles and a given line



Given the line (c) and the angles A and B .

In the constructed triangle ABC , AB is equal to line (c) and angle A and B are equal to the given angles A and B .

- (123). Theorem: The opposite sides of a parallelogram are equal, and the opposite angles are equal.



Given the parallelogram $ABCD$.

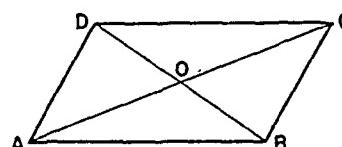
Side AB is equal to DC and AD is equal to BC , also angle D equals angle B and angle DAB equals angle DCB .

Corollary: A diagonal divides a parallelogram into two congruent triangles.

Corollary: Segments of parallel lines cut off by parallel lines are equal.

Corollary: Two parallel lines are everywhere equidistant.

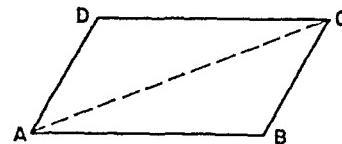
- (124). The diagonals of a parallelogram bisect each other.



Given parallelogram $ABCD$ with the diagonals AC and BD intersecting at O .

AO equals OC , and DO equals OB .

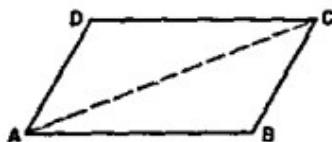
- (125). Theorem: If the two pairs of opposite sides of a quadrilateral are equal, the figure is a parallelogram.



Given the quadrilateral with AB equal to DC and AD equal to BC .
Quadrilateral $ABCD$ is a parallelogram.

- (126). Theorem: If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

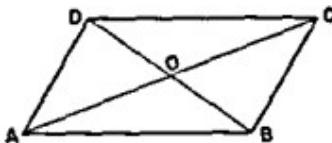
SOLUTION OF EQUATIONS



Given the quadrilateral $ABCD$ in which DC is both equal and parallel to AB .

Quadrilateral $ABCD$ is a parallelogram.

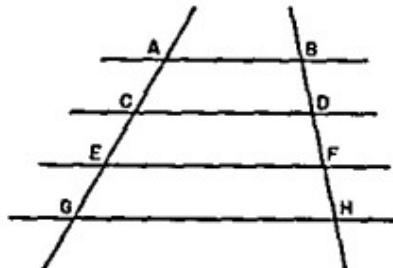
- (127). Theorem. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



Given the quadrilateral $ABCD$ with its diagonals AC and BD intersecting at O , so that AO equals OC and DO equals OB .

Quadrilateral $ABCD$ is a parallelogram.

- (128). Theorem. If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.

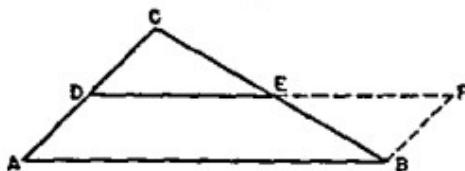


Given the parallel lines AB , CD , EF , and GH intercepting the equal segments AC , CE , and EG on the transversal AG .

Segments BD , DF , and FH are equal.

Corollary. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.

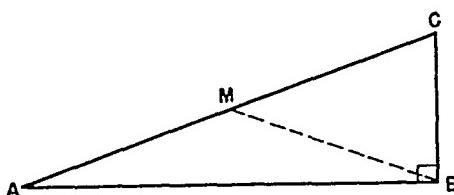
- (129). Theorem. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to one-half the third side.



Given the triangle ABC in which D is the midpoint of AC and E is the midpoint of BC .

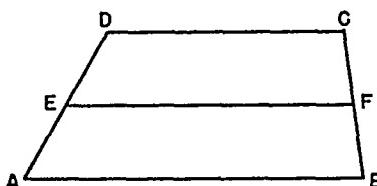
DE is parallel to AB , and DE is equal to one-half AB .

- (130). Theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.



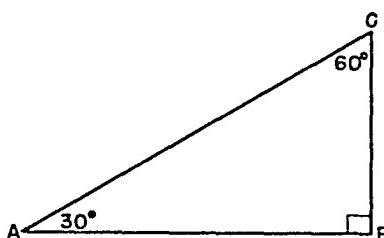
Given right triangle ABC with M the midpoint of AC .
 MA , MB , and MC are equal.

- (131). Theorem: The median of a trapezoid is parallel to the bases and equal to one-half their sum.



Given EF the median of the trapezoid $ABCD$.
Median EF is parallel to AB and DC , also EF is equal to one-half the sum of AB plus DC .

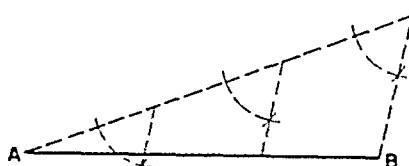
- (132). Theorem: In a 30° - 60° right triangle, the side opposite the 30° angle is equal to one-half the length of the hypotenuse.



Given right triangle ABC with angle A equal to 30° and angle C equal to 60° .

Side BC is equal to one-half the length of AC .

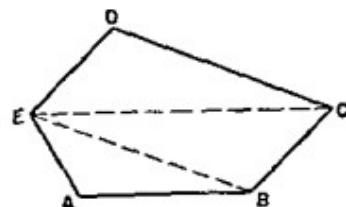
- (133). Problem: To divide a given line segment into any number of equal parts.



Given the line AB .

Line AB is divided into three equal parts.

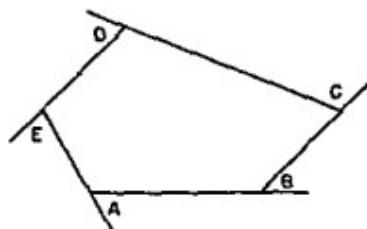
- (134). Theorem. The sum of the interior angles of a polygon of N sides is $(N - 2)$ times 180° .



Given polygon $ABCDE$, any polygon having N sides.

The sum of the interior angle A, B, C, D , and E is equal to $(N - 2)$ times 180° .

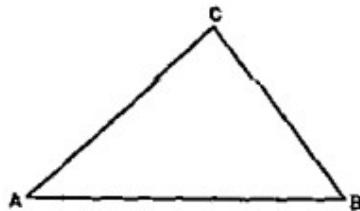
- (135) Theorem The sum of the exterior angles of a polygon, formed by extending the sides in succession, is equal to 360° .



Given the polygon $ABCDE$, any polygon of N sides

The sum of the exterior angles A, B, C, D , and E is equal to 360° .

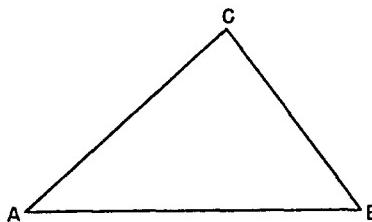
- (136). Theorem If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.



Given triangle ABC with side AC greater than side BC .

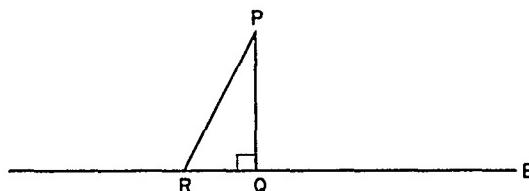
Angle B is greater than angle A .

- (137). Theorem: If two angles of a triangle are unequal, the sides opposite these angles are unequal and the greater side is opposite the greater angle.



Given the triangle ABC with angle B greater than angle A .
Side AC is greater than side BC .

- (138). Theorem: The perpendicular is the shortest line that can be drawn from a given point to a given straight line.

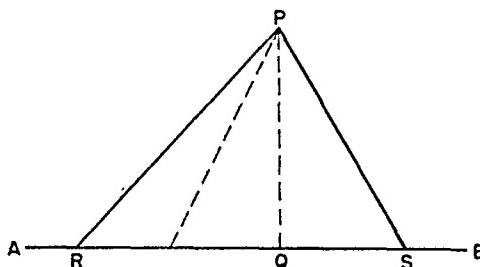


Given the point P and the line AB . PQ is perpendicular to AB , and PR is any other line from P to AB .

Line PQ is less than PR .

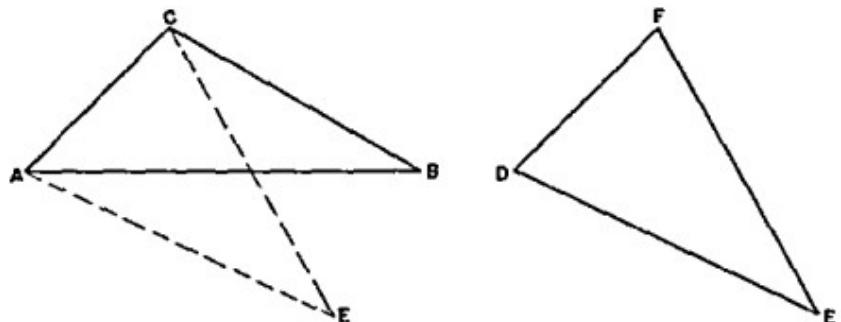
Corollary: If a line is the shortest line that can be drawn from a given point to a given line, it is the perpendicular from the point to the line.

- (139). Theorem: If two oblique straight lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, the more remote is the greater.



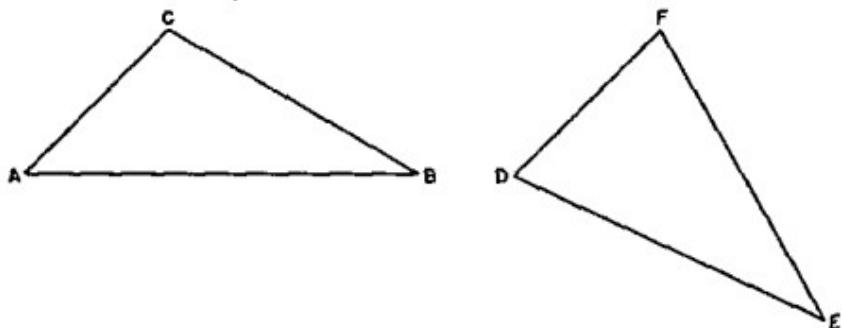
Given the point P outside the line AB . PQ is perpendicular to AB , and QR is greater than QS .
 PR is greater than PS .

- (140). Theorem: If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first triangle greater than the included angle of the second, then the third side of the first triangle is greater than the third side of the second.



Given the two triangles ABC and DEF with AC equal to DF , CB equal to FE , and angle ACB greater than angle F .
 AB is greater than DE .

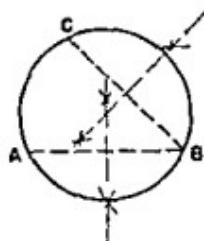
- (141). Theorem. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.



Given the two triangles ABC and DEF with AC equal to DF , CB equal to FE , and AB greater than DE .
Angle C is greater than angle F .

The Circle

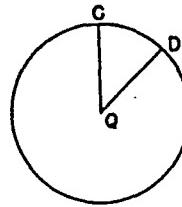
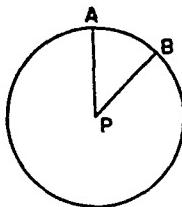
- (142). Theorem: Through any three given points not lying in a straight line one circle, and only one, can be drawn



Given points A , B , and C , not in a straight line.
Only the one circle may be drawn through points A , B , and C .

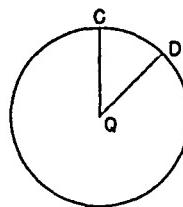
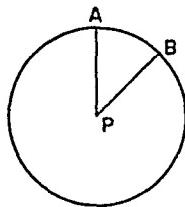
Corollary: The center of a circle circumscribed about a right triangle is the midpoint of the hypotenuse.

- (143). **Theorem:** In the same circle, or in equal circles, equal central angles intercept equal arcs.



Given the equal circles P and Q with angle APB equal to angle CQD .
Arc AB is equal to arc CD .

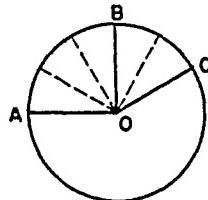
- (144). **Theorem:** In the same circle, or in equal circles, equal arcs determine equal central angles.



Given the equal circles P and Q with arc AB equal to arc CD .
Angle APB is equal to angle CQD .

Corollary: If in the same circle, or in equal circles, two central angles are unequal, the greater angle intercepts the greater arc; and
Conversely, if two arcs are unequal, the greater arc determines the greater angle at the center.

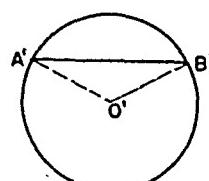
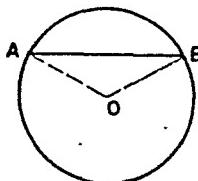
- (145). **Theorem:** In the same circle, or in equal circles, two central angles have the same ratio as their intersected arcs.



Given central angles AOB and BOC .

Angle AOB divided by angle BOC is equal to arc AB divided by arc BC .

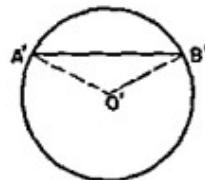
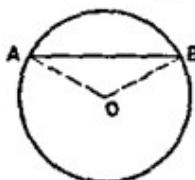
- (146). **Theorem:** In the same circle, or in equal circles, equal chords determine equal minor arcs and equal major arcs.



SOLUTION OF EQUATIONS

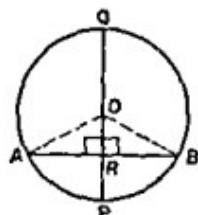
Given the equal circles O and O' with chord AB equal to chord $A'B'$.
Arc AB is equal to arc $A'B'$.

- (147). Theorem: In the same circle, or in equal circles, if two arcs are equal, their chords are equal.



Given the equal circles O and O' with arc AB equal to arc $A'B'$.
Chord AB is equal to chord $A'B'$.

- (148) Theorem. A diameter perpendicular to a chord of a circle bisects the chord and the arcs determined by the chord.



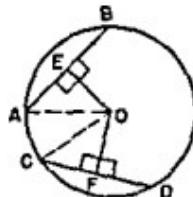
Given the circle O , and the diameter PQ perpendicular to the chord AB at point R

AR is equal to RB , also arc AP is equal to arc PB , and arc AQ is equal to arc BQ .

Corollary A diameter bisecting a chord, which is not a diameter, is perpendicular to the chord

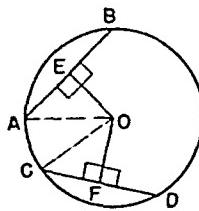
Corollary The perpendicular bisector of a chord passes through the center of the circle.

- (149). Theorem In the same circle, or in equal circles, equal chords are equidistant from the center.



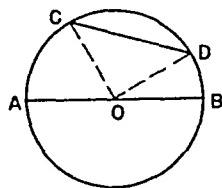
Given the circle O with chord AB equal to chord CD . OE is perpendicular to AB , and OF is perpendicular to CD .
 OE is equal to OF .

- (150). Theorem: In the same circle, or in equal circles, chords equidistant from the center are equal.



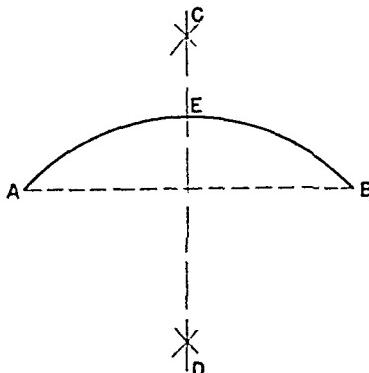
Given the circle O with chords AB and CD . OE is perpendicular to AB , OF is perpendicular to CD , and OE is equal to OF .

- (151). Theorem: The diameter of a circle is greater than any other chord.



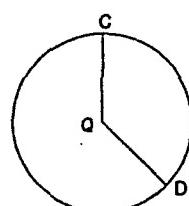
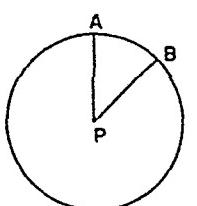
Given the diameter AB and any other chord CD in the circle O .
Diameter AB is greater than chord CD .

- (152). Problem: To bisect a given circular arc.



Given the circular arc AB .
 CD bisects arc AB at point E .

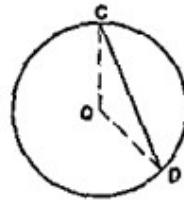
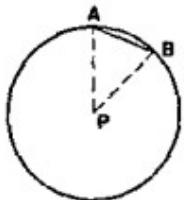
- (153). Theorem: In the same circle, or in equal circles, if two central angles are unequal, their arcs are unequal and the greater central angle has the greater arc.



Given the equal circles P and Q with angle CQD greater than angle APB .
 Arc CD is greater than arc AB .

Corollary: Conversely, in the same circle or in equal circles, if two arcs are unequal, they have unequal central angles, and the greater arc has the greater central angle.

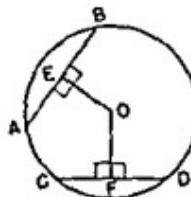
- (154). Theorem: In the same circle, or in equal circles, if two chords are unequal, the greater chord determines the greater arc.



Given the equal circles P and Q with chord CD greater than chord AB .
 Arc CD is greater than arc AB .

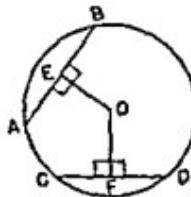
Corollary. Conversely, in the same circle or in equal circles, if two arcs are unequal, the greater arc determines the greater chord.

- (155). Theorem: In the same circle, or in equal circles, if two chords are unequal, the shorter is at the greater distance from the center.



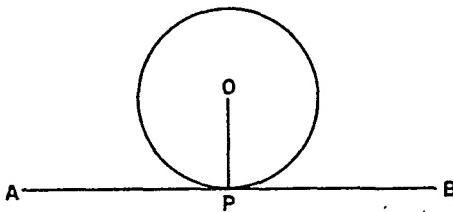
Given the circle O and the chord CD less than chord AB . OE is perpendicular to AB , and OF is perpendicular to CD .
 OF is greater than OE .

- (156). Theorem. In the same circle, or in equal circles, if two chords are unequally distant from the center, the chord at the greater distance from the center is the shorter.



Given the circle O and the chords AB and CD . OE is perpendicular to AB , OF is perpendicular to CD , and OF is greater than OE .
 Chord CD is less than chord AB .

- (157). Theorem: If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.



Given AB tangent to the circle O at the point P . OP is a radius.

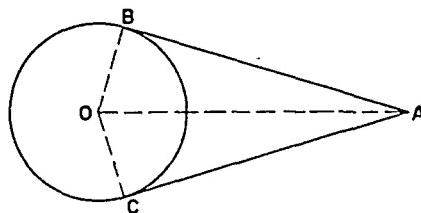
Tangent AB is perpendicular to the radius OP .

Corollary: A straight line perpendicular to a radius at its outer extremity is tangent to the circle.

Corollary: A perpendicular to a tangent at the point of contact passes through the center of the circle.

Corollary: A perpendicular drawn from the center of a circle to a tangent passes through the point of contact.

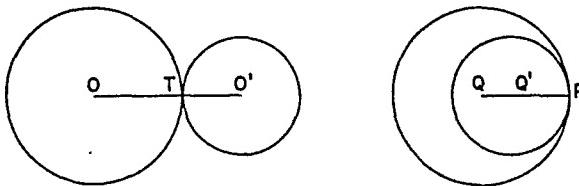
- (158). Theorem: The tangents to a circle from an external point are equal and make equal angles with the line joining that point to the center.



Given the tangents AB and AC from the point A outside the circle O , and AO a line from A to the center O .

Tangent AB is equal to tangent AC , and angle BAO is equal to angle CAO .

- (159). Theorem: If two circles are tangent to each other externally or internally, the line of centers passes through the point of tangency.



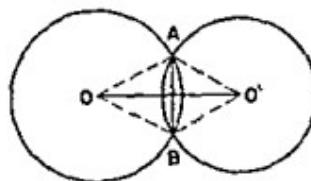
Case I. Given the circles O and O' tangent externally at T .

The line of centers OO' passes through point T .

Case II. Given the circles Q and Q' tangent internally at P .

The line of centers QQ' passes through point P .

- (160). Theorem: If two circles intersect, the line of centers is the perpendicular bisector of their common chord.

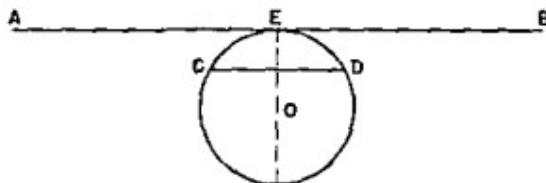


Given the circles O and O' , intersecting at A and B , with AB the common chord, and OO' the line of centers.

Line of centers OO' is the perpendicular bisector of the chord AB .

(161). Theorem Parallel lines intercept equal arcs on a circle.

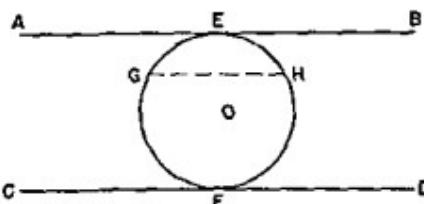
Case I If one line is a tangent and the other is a chord.



Given AB tangent at E to circle O , and parallel to chord CD .

Arc CE is equal to arc DE .

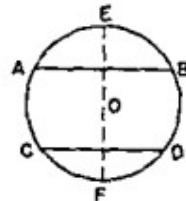
Case II If both lines are tangents



Given the circle O with AB , the tangent at E , parallel to CD , the tangent at F .

Arc EGF is equal to arc EHF .

Case III. If both lines are chords.

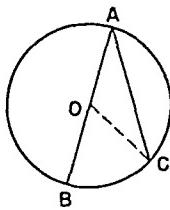


Given the circle O and the chord AB parallel to the chord CD .

Arc AC is equal to arc BD .

(162). Theorem: An inscribed angle is measured by one-half its intercepted arc.

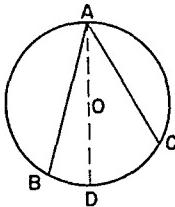
Case I. If one side of the angle is a diameter.



Given angle BAC inscribed in the circle O , and AB passing through the center O .

Angle BAC is measured by one-half arc BC .

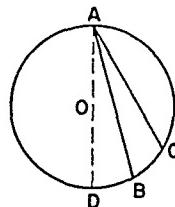
Case II. If the center of the circle lies within the inscribed angle.



Given the inscribed angle BAC , with the center of the circle O lying within the angle.

Angle BAC is measured by one-half arc BC .

Case III. If the center of the circle lies outside the inscribed angle.



Given the inscribed angle BAC , with the center O outside the angle.

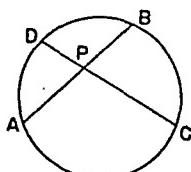
Angle BAC is measured by one-half arc BC .

Corollary: All angles inscribed in the same segment or in equal segments, are equal.

Corollary: An angle inscribed in a semicircle is a right angle, or 90° .

Corollary: An angle inscribed in a segment whose arc is greater than a semicircle is an acute angle. An angle inscribed in a segment whose arc is less than a semicircle is an obtuse angle.

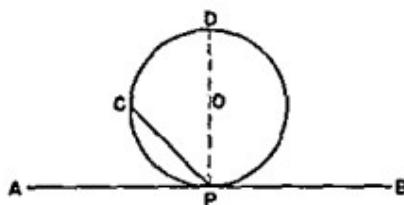
- (163). Theorem: An angle formed by two chords intersecting within a circle is measured by half the sum of its intercepted arc and that of its vertical angle.



Given chords AB and CD intersecting at the point P .

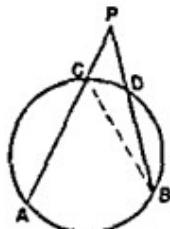
Angle BPC is measured by one-half the sum of arc BC and arc AD .

- (164). Theorem: The angle formed by a tangent and a chord drawn from the point of tangency is measured by half of its intercepted arc.

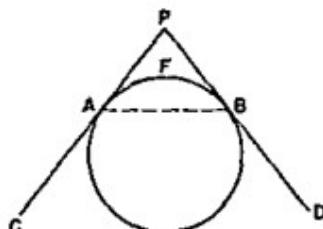


Given AB a tangent to the circle O at the point P ; PC a chord.
Angle APC is measured by one-half arc PC .

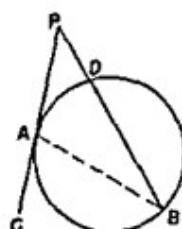
- (165). Theorem: An angle formed by two secants of a circle, or by two tangents, or by a secant and a tangent, intersecting at a point outside the circle, is measured by half the difference between the intercepted arcs.



Case I



Case II



Case III

- I. Given the circle $ACDB$, and the angle APB formed by the secants PA and PB , meeting at the point P outside the circle, and intersecting the circle at C and D , respectively.

Angle P is measured by half the difference between arc AB and arc CD .

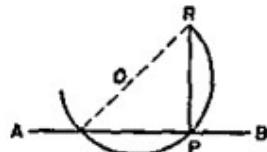
- II. Given the angle APB formed by the tangents PC and PD touching the circle AEB at points A and B , respectively.

Angle P is measured by half the difference between arc AEB and arc AFB .

- III. Given the circle ADB , and the angle CPB formed by the tangent PG touching the circle at point A and the secant PB intersecting the circle at D .

Angle P is measured by half the difference between arc AB and arc AD .

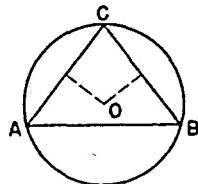
- (166). Problem Construct a perpendicular to a given line at a given point.



Given the point P in the line AB .

RP is perpendicular to AB at point P .

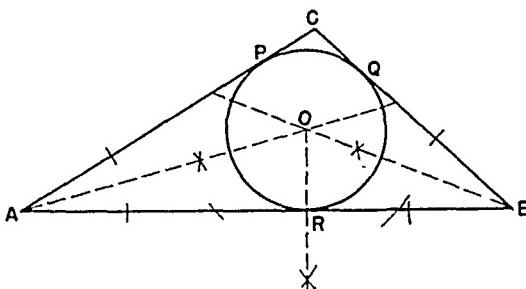
- (167). Problem: To circumscribe a circle about a given triangle



Given the triangle ABC .

The circumscribed circle O passes through the vertices A , B , and C .

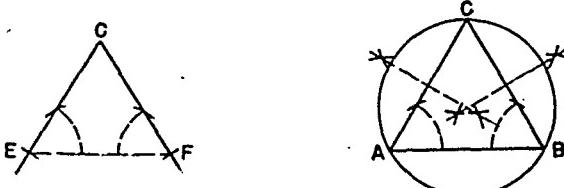
- (168). Problem: To inscribe a circle in a given triangle.



Given the triangle ABC .

The inscribed circle O is tangent at three points, P , Q , and R , to the sides of the triangle ABC .

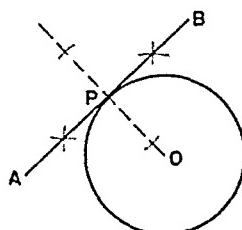
- (169). Problem: On a given straight line as a chord to construct a circular segment in which a given angle may be inscribed.



Given the straight line AB and the angle C .

The circular segment ACB constructed on chord AB contains the given angle C .

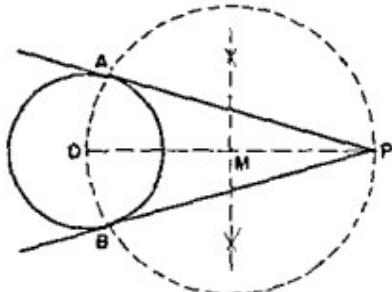
- (170). Problem: To construct a tangent to a given circle at a point on the circle.



Given the point P on the circle O .

Line APB is tangent to circle O through point P .

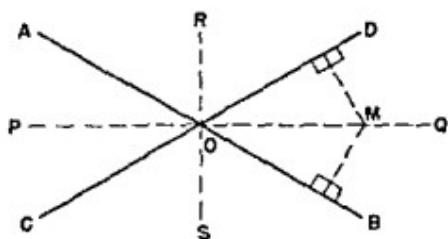
- (171) Problem. To construct a tangent to a given circle from a given external point



Given P , any point outside the circle O

PA and PB are both tangents to circle O from point P .

- (172). Theorem The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.

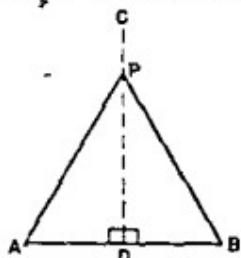


Given PQ and RS the bisectors of the angles formed by the intersecting lines AB and CD

The pair of lines PQ and RS is the locus of points equidistant from AB and CD

Corollary The locus of points within an angle equidistant from the sides is the line that bisects the angle

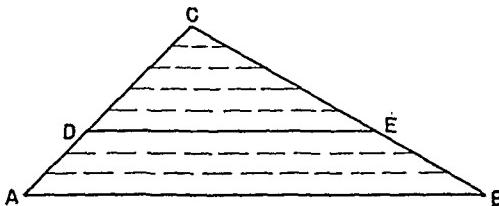
- (173) Theorem The locus of points equidistant from two given points is the perpendicular bisector of the line joining them



Given the two points A and B , and CD the perpendicular bisector of AB . Line CD is the locus of points equidistant from A and B .

Proportion—Similar Figures

- (174). Theorem: A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.



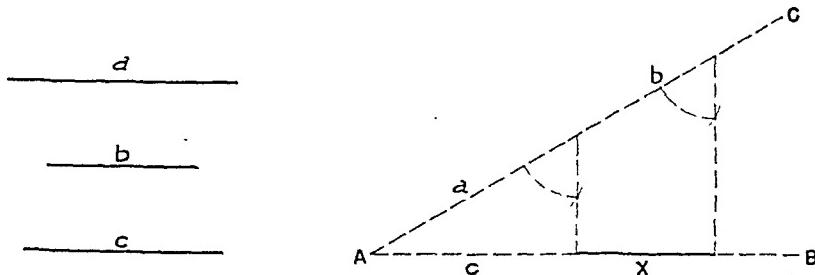
Given the triangle ABC with DE parallel to AB and intersecting the sides AC and CB .

AD is to DC as BE is to EC .

Corollary: Segments cut off on two transversals by a series of parallel lines are proportional.

Corollary: When two sides of a triangle are cut by a line parallel to the base, one side is to either of its segments as the other side is to its corresponding segment.

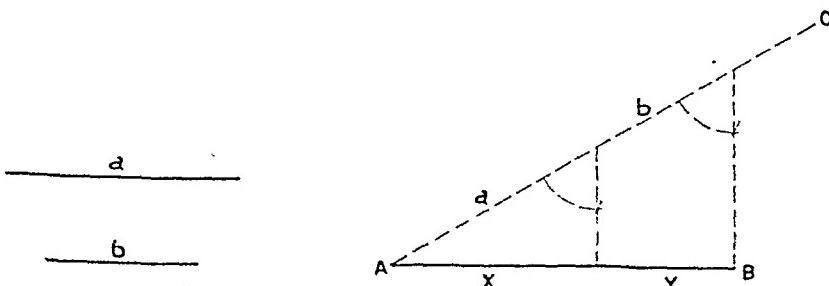
- (175). Problem: To construct the fourth proportional to three given line-segments.



Given the line segments (a) , (b) , and (c) .

Line segment (a) is to (b) as (c) is to (x) .

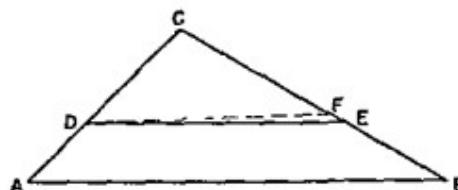
- (176). Problem: To divide a given line segment into parts proportional to two given line segments.



Given the line segment AB and the two lines (a) and (b) .

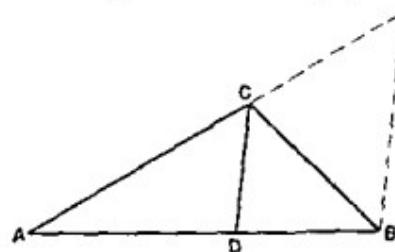
Line AB is divided so that (a) is to (b) as (x) is to (y) .

- (177) Theorem. If a line divides two sides of a triangle proportionally, it is parallel to the third side.



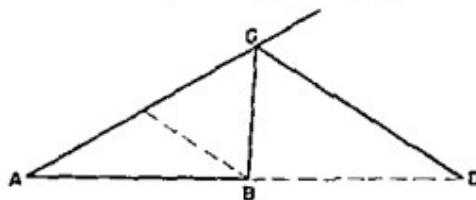
Given the triangle ABC and the line DE intersecting CA and CB so that
 CA is to CD as CB is to CE
 DE is parallel to AB

- (178) Theorem The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides



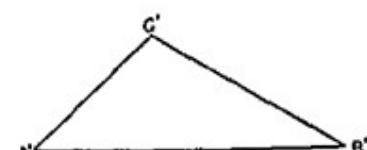
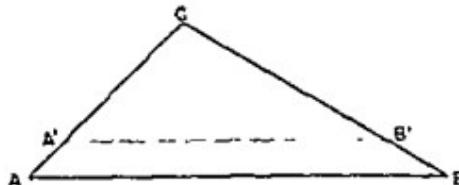
Given the triangle ABC with CD bisecting the angle ACB , and meeting AB at D
 AD is to DB as AC is to CB

- (179) Theorem The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides



Given the triangle ABC with CD the bisector of the exterior angle at C .
 AD is to BD as AC is to BC .

- (180). Theorem. If two triangles have the angles of one respectively equal to the angles of the other, the triangles are similar



Given the triangles ABC and $A'B'C'$ in which angle A equals angle A' , angle B equals angle B' , and angle C equals angle C' .

Triangles ABC and $A'B'C'$ are similar.

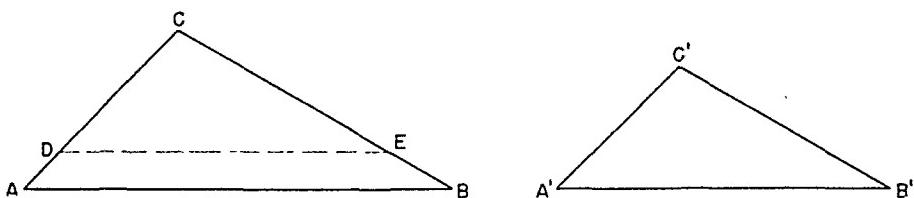
Corollary: If two triangles have two angles of one respectively equal to two angles of the other, the triangles are similar.

Corollary: If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.

Corollary: If each of two triangles is similar to a third triangle, they are similar to each other.

Corollary: If a line parallel to one side of a triangle cuts the other two sides, a triangle is formed which is similar to the given triangle.

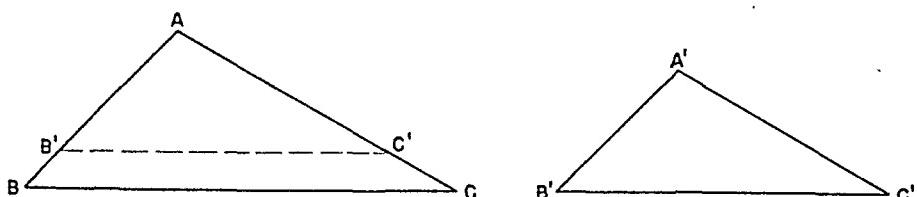
- (181). Theorem: If the corresponding sides of two triangles are proportional, the triangles are similar.



Given the triangles ABC and $A'B'C'$ in which AB is to $A'B'$ as AC is to $A'C'$ as BC is to $B'C'$.

Triangles ABC and $A'B'C'$ are similar.

- (182). Theorem: If two triangles have an angle of one equal to an angle of the other, and the sides including these angles proportional, the triangles are similar.

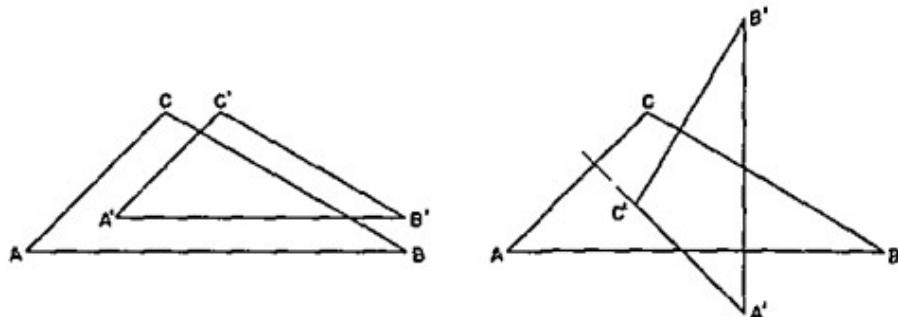


Given the triangles ABC and $A'B'C'$ in which angle A is equal to angle A' , and AB is to $A'B'$ as AC is to $A'C'$.

Triangles ABC and $A'B'C'$ are similar.

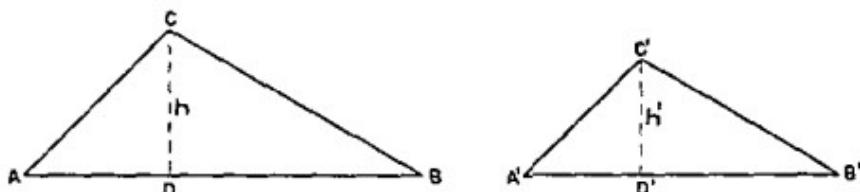
Corollary: Two isosceles triangles in which the vertex angles are equal are similar.

- (183). Theorem: Two triangles are similar if their corresponding sides are either parallel or perpendicular to each other.



Given the triangles ABC and $A'B'C'$ in which AB , BC and AC are parallel or perpendicular respectively to $A'B'$, $B'C'$, and $A'C'$.
Triangles ABC and $A'B'C'$ are similar.

- (184). Theorem: The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.



Given the similar triangles ABC and $A'B'C'$ with (h) and (h') corresponding altitudes

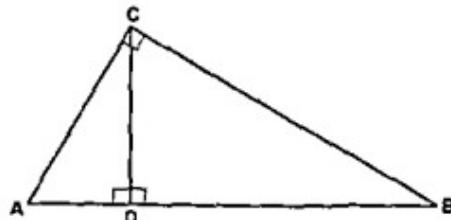
Altitude (h) is to (h') as AC is to $A'C'$.

- (185). Theorem: If, in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse:

I. The two triangles thus formed are similar to the given triangle and similar to each other

II. The perpendicular is the mean proportional between the segments of the hypotenuse

III. Each leg of the given triangle is the mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to that leg.



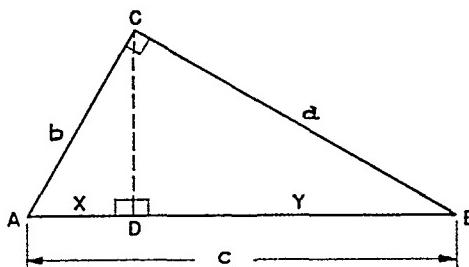
Given the right triangle ACB with the altitude CD cutting off segments AD and DB on hypotenuse AB .

I. Triangles ADC , ACB , and CDB are similar.

II. AD is to CD as CD is to DB .

III. AB is to AC as AC is to AD , and AB is to CB as CB is to DB .

- (186). Theorem: In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

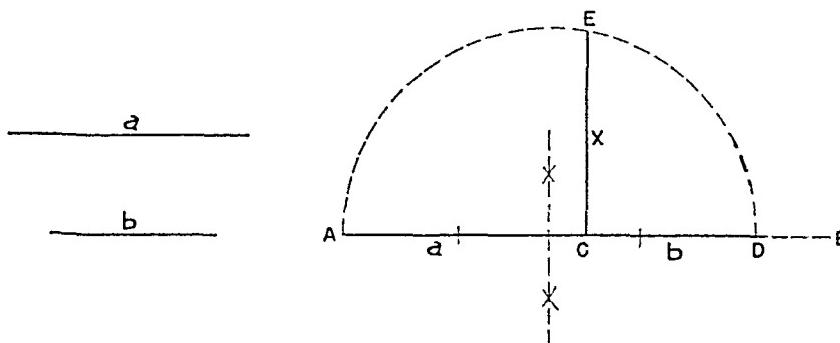


Given the right triangle ACB with (c) the hypotenuse and (a) and (b) the legs.

The hypotenuse $(c)^2$ is equal to the sum of $(a)^2$ and $(b)^2$.

Corollary: The square of either leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.

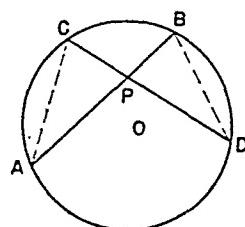
- (187). Problem: To construct the mean proportional between two given line segments.



Given the straight lines (a) and (b).

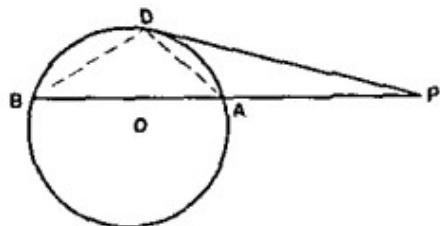
Perpendicular EC is the required mean proportional.

- (188). Theorem: If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.



Given the two chords AB and CD intersecting at P within the circle O .
Segment CP is to PD as AP is to PB .

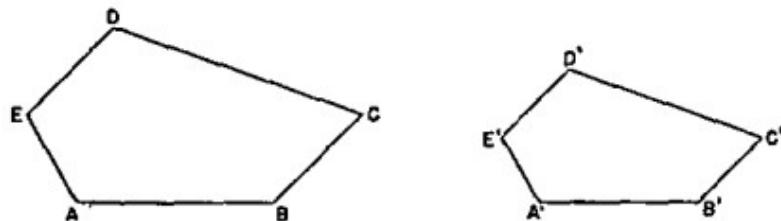
- (189). Theorem. If, from a point outside a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the whole secant and its external segment.



Given the circle O with point P outside the circle, PD a tangent, and PB a secant with AP its external segment.

PB is to PD as PD is to AP

- (190) Theorem. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

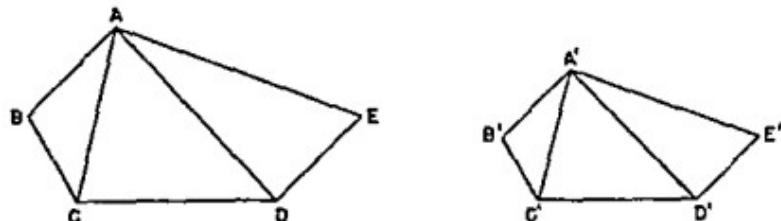


Given the similar polygons $ABCDE$ and $A'B'C'D'E'$ with the perimeters denoted by P and P' .

P is to P' as AB is to $A'B'$.

Corollary The perimeters of two similar polygons have the same ratio as any two corresponding diagonals

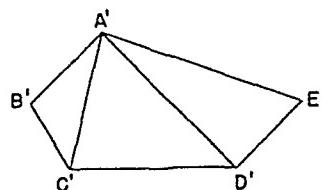
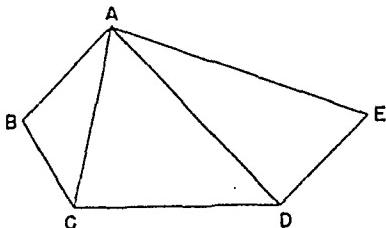
- (191). Theorem. If two polygons are similar, they can be divided into the same number of triangles, each similar to each other and similarly placed.



Given the similar polygons $ABCDE$ and $A'B'C'D'E'$ with diagonals drawn from corresponding vertices A and A' .

Each triangle in one polygon is similar to its corresponding triangle in the other polygon.

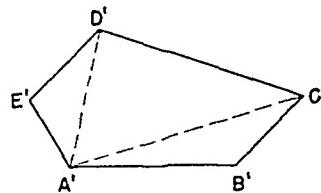
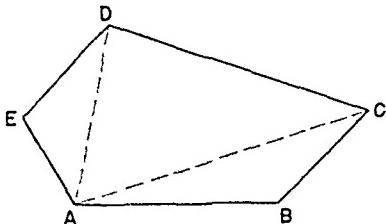
- (192). Theorem: If two polygons are composed of the same number of triangles each similar to each other and similarly placed, the polygons are similar.



Given the polygons $ABCDE$ and $A'B'C'D'E'$ in which the corresponding triangles are similar and similarly placed.

Polygon $ABCDE$ is similar to polygon $A'B'C'D'E'$.

- (193). Problem: Upon a given line segment as a side, to construct a polygon similar to a given polygon.



Given a polygon $ABCDE$ and a line $A'B'$ corresponding to the side AB .
Polygon $A'B'C'D'E'$ constructed upon side $A'B'$ is similar to polygon $ABCDE$.

Areas of Polygons

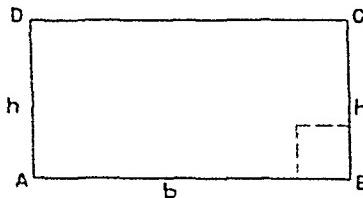
- (194). Theorem: The areas of two rectangles having equal altitudes are to each other as their bases.

Corollary: The areas of two rectangles having equal bases are to each other as their altitudes.

Corollary: Two rectangles are equal, if they have equal bases and equal altitudes.

- (195). Theorem: The areas of two rectangles are to each other as the products of their bases and their altitudes.

- (196). Theorem: The area of a rectangle is equal to the product of its base and its altitude.

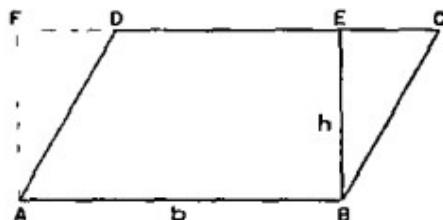


UNIT

SOLUTION OF EQUATIONS

Given the rectangle $ABCD$, with the base containing (b) , and the altitude containing (b) units of linear measure.
The area of R is equal to the product of (b) and (b) .

- (197). Theorem: The area of a parallelogram is equal to the product of its base and its altitude.



Given the parallelogram $ABCD$ with the base AB equal to (b) and its altitude BE equal to (b)

The area of the parallelogram $ABCD$ is equal to the product of (b) and (b)

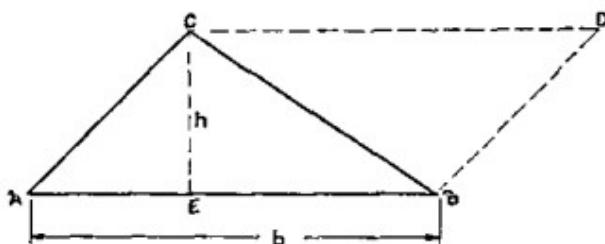
Corollary. Parallelograms with equal bases and equal altitudes are equal in area.

Corollary. The areas of two parallelograms are to each other as the products of their bases and their altitudes

Corollary. The areas of two parallelograms with equal bases are to each other as their altitudes, and

The areas of two parallelograms with equal altitudes are to each other as their bases.

- (198) Theorem: The area of a triangle is equal to half the product of its base and its altitude



Given the triangle ABC with its altitude CE equal to (b) and its base AB equal to (b) .

The area of triangle ABC is equal to one-half the product of (b) and (b) .

Corollary: The area of a triangle is equal to one-half that of a parallelogram having the same base and the same altitude.

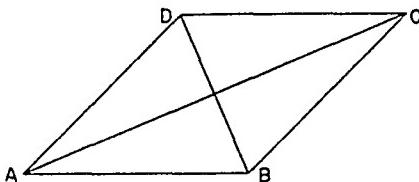
Corollary: Two triangles which have equal bases and equal altitudes (or equal bases in the same straight line and their vertices in a line parallel to the base) are equal in area.

Corollary: Two triangles which have equal bases are to each other as their altitudes; and

Two triangles which have equal altitudes are to each other as their bases.

Corollary: Any two triangles are to each other as the products of their bases and their altitudes.

- (199). Theorem: The area of a rhombus is equal to half the product of the diagonals of the rhombus.

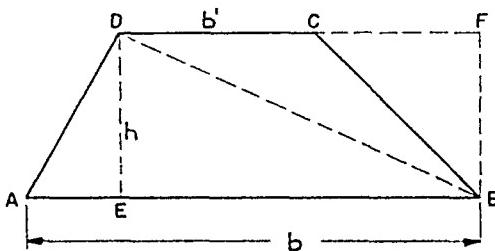


Given the rhombus $ABCD$ with the diagonals AC and DB .

The area of rhombus $ABCD$ is equal to one-half the product of AC and DB .

Corollary: If the diagonals of a quadrilateral are perpendicular to each other, the area of the quadrilateral is equal to half the product of the diagonals.

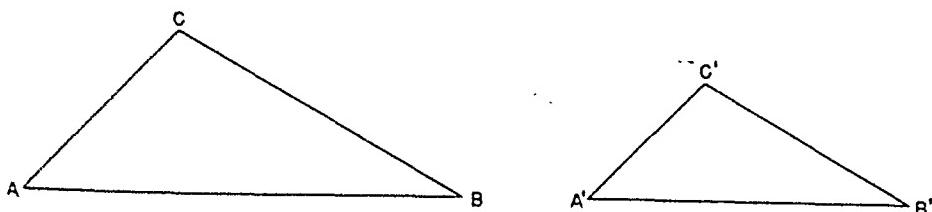
- (200). Theorem: The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.



Given the trapezoid $ABCD$ with altitude DE equal to (h) , its base AB equal to (b) and its base DC equal to (b') .

The area of trapezoid $ABCD$ is equal to one-half the product of (h) times the sum of the bases, (b) and (b') .

- (201). Theorem: The areas of two similar triangles are to each other as the squares of any two corresponding sides.

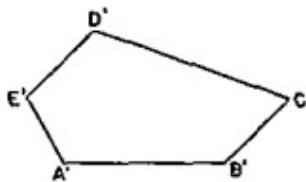
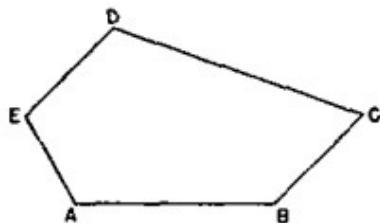


SOLUTION OF EQUATIONS

Given the similar triangles ABC and $A'B'C'$ with AB and $A'B'$ corresponding sides.

Triangle ABC is to triangle $A'B'C'$ as the square of AB is to the square of $A'B'$.

- (202). Theorem. The areas of two similar polygons are to each other as the squares of any two corresponding sides.



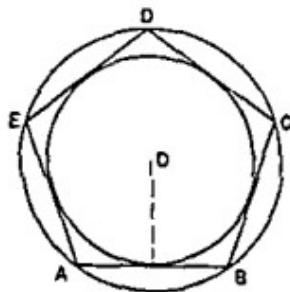
Given two similar polygons $ABCDE$ and $A'B'C'D'E'$ with AB and $A'B'$ corresponding sides

Polygon $ABCDE$ is to $A'B'C'D'E'$ as the square of side AB is to the square of $A'B'$.

Corollary The areas of two similar polygons are to each other as the squares of their perimeters, or as the squares of any two corresponding lines.

Regular Polygons—Measurement of the Circle

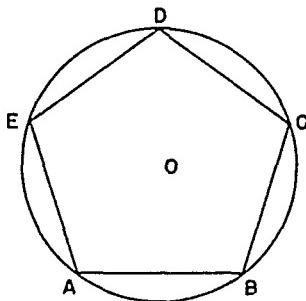
- (203) Theorem A circle can be circumscribed about any regular polygon, and a circle can also be inscribed in any regular polygon.



Given the regular polygon $ABCDE$.

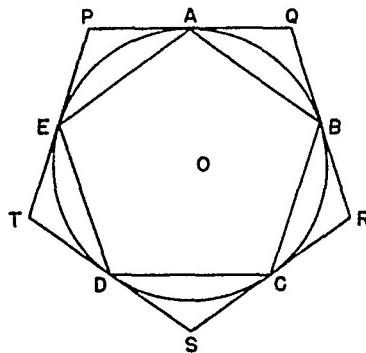
The circumscribed circle O passes through the vertices A, B, C, D, E , and the inscribed circle O touches all the sides of the given polygon.

- (204). Theorem: An equilateral polygon inscribed in a circle is a regular polygon.



Given the equilateral polygon $ABCDE$ inscribed in the circle O .
 $ABCDE$ is a regular polygon.

- (205). Theorem: If a circle is divided into any number of equal arcs,
I. The chords of these arcs form a regular inscribed polygon.



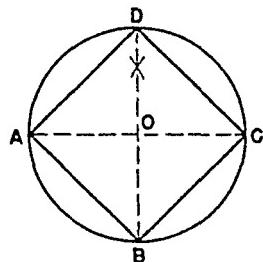
II. The tangents to the circle at the points of division form a regular circumscribed polygon.

Given the circle O divided into the equal arcs AB , BC , CD , DE , and EA , the chords AB , BC , etc., and the tangents PQ , QR , etc., drawn at the points A , B , etc.

$ABCDE$ is a regular inscribed polygon, and $PQRST$ is a regular circumscribed polygon.

Corollary: An equiangular polygon circumscribed about a circle is a regular polygon.

- (206). Problem: To inscribe a square in a given circle.



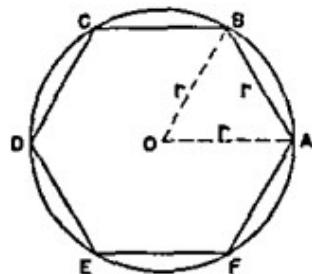
Given the circle O .

SOLUTION OF EQUATIONS

$ABCD$ is the required inscribed square in circle O .

Corollary: By bisecting the arcs AB , BC , CD , and DA and drawing chords, a regular polygon of eight sides is inscribed.

- (207) Problem. To inscribe a regular hexagon in a given circle.
Given the circle O

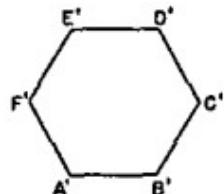
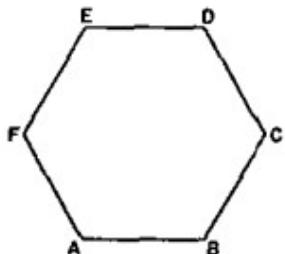


$ABCDEF$ is the regular hexagon inscribed in the circle O .

Corollary: Each side of a regular inscribed hexagon is equal to the radius of the circle.

Corollary. By joining the alternate vertices of a regular inscribed hexagon, an equilateral triangle can be inscribed in a given circle.

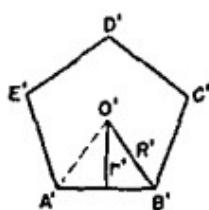
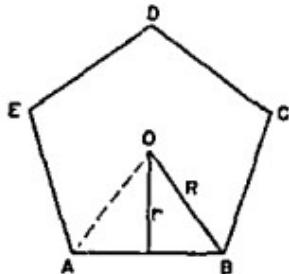
- (208). Theorem Regular polygons of the same number of sides are similar.



Given the regular polygons $ABCDEF$ and $A'B'C'D'E'F'$ with the same number of sides

The regular polygons $ABCDEF$ and $A'B'C'D'E'F'$ are similar.

- (209). Theorem. The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles (apothems).

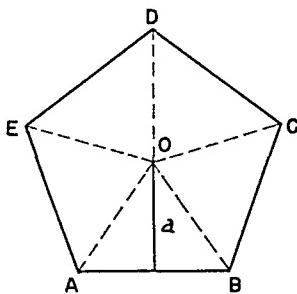


Given two regular polygons of (n) sides, of which P and P' are the perimeters, R and R' the radii of the circumscribed circles, and r and r' the radii of the inscribed circles.

P is to P' as R is to R' as r is to r' .

Corollary: The areas of regular polygons of the same number of sides are to each other as the squares of the radii of the circumscribed, or inscribed, circles.

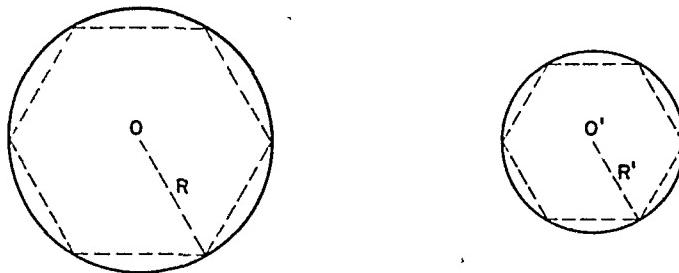
- (210). Theorem: The area of a regular polygon is equal to half the product of its apothem and its perimeter.



Given the regular polygon $ABCDE$ with perimeter (p) and apothem (a).

The area of polygon $ABCDE$ is equal to half the product of (a) and (p).

- (211). Theorem: The circumferences of two circles are to each other as their radii.

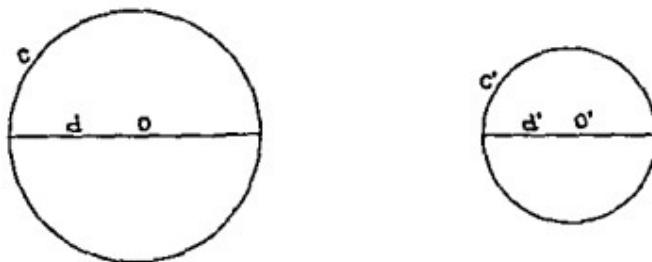


Given the circles O and O' , with circumferences C and C' and radii R and R' respectively.

C is to C' as R is to R' .

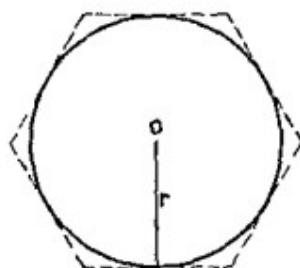
Corollary: The circumferences of two circles are to each other as their diameters.

- (212). Theorem: The ratio of the circumference of any circle to its diameter is a constant, symbolized by π , the approximate value of which is 3.1416.



Given the circle O with circumference C and diameter D . Circle O' is any other circle with circumference C' and diameter D' .
 C is to D as C' is to D' .

- (213). Theorem The area of a circle is equal to half the product of its circumference and its radius



Given the circle O with circumference C , radius R , and area A .
Area A is equal to one-half the product of R and C .
Corollary The area of a circle is equal to the square of its radius multiplied by π .
Since, $C = 2\pi R$, $A = \frac{1}{2}(2\pi R)R$, or $A = \pi R^2$.
Corollary The areas of two circles are to each other as the squares of their radii or as the squares of their diameters

$$\frac{A}{A'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2} = \frac{D'^2}{D'^2}$$

Section III

TRIGONOMETRY

INTRODUCTION

Trigonometry is a branch of mathematics that deals with the measurements of angles and the sides of triangles. This subject involves the use of numbers the same as arithmetic, equations the same as algebra, and also many facts developed in geometry. In *plane* trigonometry these relations are applied to the solution of plane triangles which may be defined as a closed figure of three straight sides all of which lie in one plane. In spherical trigonometry the three sides are not straight lines and do not lie in the same plane. The term *plane* will not be repeated in the paragraphs which follow but should be understood to apply in all cases as only plane triangles are being considered.

Types of Triangles

The definitions and methods of solution which follow apply to the following types of triangles:

Right triangles, or those triangles in which one of the interior angles is a right angle, that is 90° . The side of the triangle opposite the right angle is the hypotenuse.

Oblique triangles, or those triangles in which none of the interior angles is a right angle. Such triangles may or may not contain one angle greater than 90° .

Elements of Triangles

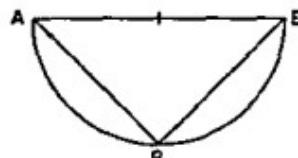
A triangle consists of six (6) elements—three (3) sides and three (3) angles. If the three angles only (no sides) are given, there will be an infinite number of triangles of the same shape but varying areas which will satisfy the stated conditions. Such triangles are called similar triangles. Consequently, the values of the sides cannot be found. At least one side must be known, and furthermore, the total number of unknown elements must not exceed three. Any triangle, either right or oblique, is said to be completely solved when, having the necessary number of elements given, the remaining elements have been found. In practical work the complete solution is not always necessary as often only the value of one or two of a possible total number of three unknown elements is required.

Before proceeding on the solution of any triangle, make certain that the necessary minimum of three elements are known, one of which must be a side. The next fact to be established is whether the given triangle is a right or an oblique triangle, as the latter type are usually solved by special formulas. Right triangles can be solved by the same methods as oblique triangles, but to do so is an unnecessary complication.

Identification of Right Triangles

To determine whether a given triangle is an oblique or a right triangle, apply one or more of the following tests, preferably the algebraic methods number three and four, as number one and two are graphical solutions and therefore subject to slight error.

- 1 Use a protractor to measure the angle which appears to be a right angle. If its value is 90° , the triangle is a right triangle.
- 2 Using a compass, construct a circle with its center at the mid-point of the hypotenuse (AB) and with a radius equal to one-half of its length. If the circumference passes through the vertex of the third angle of the triangle, then that angle (R) is a right angle



- 3 In any triangle the sum of the three interior angles is 180° . At least two of these angles must each be less than 90° . (Such angles measuring less than 90° are termed acute angles, and any angle greater than 90° is called an obtuse angle) If the value of the two acute angles A and B are known, the value of the third angle (R) may be found, by subtracting their sum ($A + B$) from 180° .

$$R = 180^\circ - (A + B)$$

If the remainder is 90° , the triangle is a right triangle.

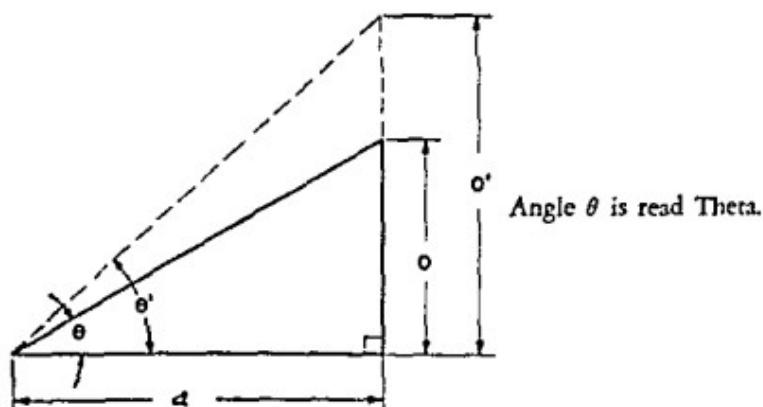
- 4 If an unknown angle happens to be 90° , this fact will soon be discovered, as the square side of the opposite (hypotenuse) this angle will equal the sum of the squares of the other two sides That is:

$$b^2 = o^2 + a^2$$

If the angle is not a right angle, the value of b^2 will not equal $o^2 + a^2$, being less if the angle is acute, and greater if the angle is obtuse

TRIGONOMETRIC FUNCTIONS IN RIGHT TRIANGLES

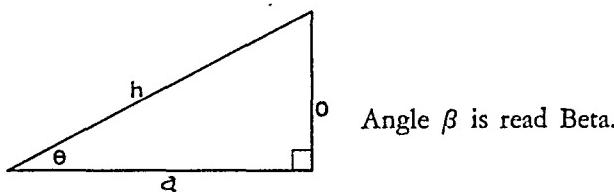
One quantity is said to be a function of another quantity if to every value of one quantity there is a corresponding value of the other. As explained below, the trigonometric functions of either of the acute angles in a right triangle are the ratios of the lengths of the various sides of the triangle.



In the triangle drawn in solid lines, o/a is the ratio of the length of the side (o), opposite angle θ , to the length of the side (a), adjacent to angle θ . If the angle at θ is changed to θ' , the value of the function o/a becomes o'/a .

Thus it is obvious that the ratio of the two sides (o) and (a) of a right triangle depends upon the size of the angle θ . Conversely, the size of the angle θ depends upon the value of the ratio of the sides (o) and (a).

As there are three sides to a triangle, it is possible to have six different ratios of sides. Each one of these ratios is named from its relation to one of the acute angles in a right triangle. Denoting the lengths of the sides of the right triangle by the letters (o) for opposite, (a) for adjacent, and (b) for hypotenuse, the various trigonometric ratios or functions for the angle θ are defined as follows:



$$\text{sine of an angle} = \frac{\text{side opposite}}{\text{hypotenuse}}; \text{ written } \sin \theta = \frac{o}{b}; \quad \sin \beta = \frac{a}{b}$$

$$\text{cosine of an angle} = \frac{\text{side adjacent}}{\text{hypotenuse}}; \text{ written } \cos \theta = \frac{a}{b}; \quad \cos \beta = \frac{o}{b}$$

$$\text{tangent of an angle} = \frac{\text{side opposite}}{\text{side adjacent}}; \text{ written } \tan \theta = \frac{o}{a}; \quad \tan \beta = \frac{a}{o}$$

$$= \frac{o}{a} = \frac{o/b}{a/b} = \frac{\sin \theta}{\cos \theta} \quad (\text{Rule 224})$$

$$\text{cotangent of an angle} = \frac{\text{side adjacent}}{\text{side opposite}}; \text{ written } \cot \theta = \frac{a}{o}; \quad \cot \beta = \frac{o}{a}$$

$$\text{secant of an angle} = \frac{\text{hypotenuse}}{\text{side adjacent}}; \text{ written } \sec \theta = \frac{b}{a}; \quad \sec \beta = \frac{b}{o}$$

$$\text{cosecant of an angle} = \frac{\text{hypotenuse}}{\text{side opposite}}; \text{ written cosec } \theta = \frac{b}{o}; \quad \text{cosec } \beta = \frac{b}{a}$$

The word cosine is an abbreviation of the expression, complement of the sine, used centuries ago. An inspection of the definitions given above shows that the sine of any angle is the same ratio, and therefore has the same numerical value, as the cosine of its complementary angle. Similarly, the cosine of any angle has the same numerical value as the sine of its complementary angle. This may be stated algebraically.

$$\sin \theta = \cos (90^\circ - \theta) \quad (\text{Rule 218})$$

$$\cos \theta = \sin (90^\circ - \theta) \quad (\text{Rule 219})$$

A similar relationship exists between the tangent and the cotangent, and between the secant and the cosecant functions.

The ratios, cotangent, secant and cosecant, are not ordinarily necessary as they are merely the reciprocals of the tangent, cosine, and sine respectively. However, these reciprocal functions may be employed to advantage if the solution is being performed by long-hand computations.

GEOMETRIC RELATIONS

The solution of any triangle is accomplished by means of the trigonometric functions and by use of certain geometric relations. The following are the more useful of the geometric relations applying to *right* triangles and also applying to *any* triangle as specified. The term any triangle is intended to include both right and oblique triangles. Additional geometric relations are contained in SECTION II—GEOMETRY.

In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides (Rule 186). This rule is often erroneously applied to oblique triangles.

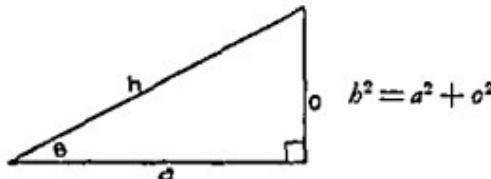
In a right angle the hypotenuse is greater than either of the two legs, but is less than their sum. In any triangle the longest side is always less than the sum of the other two sides.

In a right triangle the greater leg is opposite the greater of the two acute angles. In any triangle, equal sides are opposite equal angles, and the greatest side is opposite the greatest angle.

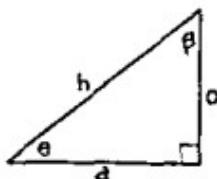
In a right triangle the sum of the two acute angles must be 90° . If one of the acute angles of a right triangle is equal to one of the acute angles of another right triangle, then the two triangles are similar, since all three of the interior angles are equal. In any triangle, the sum of the three interior angles is 180° .

The corresponding sides of any similar triangles, either right or oblique, are proportional.

The trigonometric ratios and the geometric relations are not independent of each other. For example, if two of the three sides of a right triangle are known, the third side can be found.



Also, if one of the trigonometric ratios are given, the other functions can be determined. Thus, if $\tan \theta = 3/4$, and the other ratios are required.



Let,

$$\tan \theta = o/a = 3/4$$

Let,

$$o = 3$$

Then,

$$a = 4$$

$$b^2 = o^2 + a^2 = 9 + 16 = 25$$

$$b = 5$$

Therefore,

$$\sin \theta = o/b = 3/5 \dots \text{also, } = \cos \beta$$

Therefore,

$$\cos \theta = a/b = 4/5 \dots \text{also, } = \sin \beta$$

The angles corresponding to these ratios may be found from a table of natural trigonometric functions as explained below. It must be remembered that the numbers 3, 4, and 5 as used in this particular problem are *relative* values of the sides only. The *absolute* values cannot be found unless the absolute value of one or more of the sides be given. If side (o) has an absolute value of 12, then the absolute value of (b) can be found as follows:

$$o^2 + a^2 = b^2$$

$$\tan \theta = 3/4 = \frac{12}{a}$$

$$\frac{3}{4} = \frac{12}{a}$$

$$a = \frac{(12)(4)}{(3)} = 16$$

$$12^2 + 16^2 = b^2$$

$$b^2 = 400$$

$$b = 20$$

A more direct solution makes use of the geometric principle that the corresponding sides of similar triangles are similar.

$$\begin{aligned} o/b &= 3/5 = 12/b \\ b &= \frac{(5)(12)}{(3)} = 20 \end{aligned}$$

USE OF TABLES OF NATURAL TRIGONOMETRIC FUNCTIONS— INTERPOLATION

The solution of any given right triangle makes necessary the use of a table of the natural trigonometric functions. The word natural is used to differentiate such tables from tables of logarithmic trigonometric tables.

A table of the natural trigonometric functions is a compilation of the numerical values of the more important trigonometric functions, sine, cosine, tangent, cotangent, and in some cases also the secant and cosecant, for all angles from 0° to 90° . In most cases the value of these functions are given for each degree and its subdivisions in intervals of one minute. The accuracy of the tabulated values is dependent upon the number of decimal places to which the functions have been calculated, and for ordinary purposes any table of four, five, or six places is considered to be sufficiently accurate.

The explanations and solutions of examples throughout Section III are based on data obtained from a five place table of natural trigonometric functions. However, the same procedure is employed if tables of either less or greater accuracy are employed.

If the given angle is measured in degrees only, or degrees and minutes only, the value of any one of the trigonometric functions can be read directly from the tables. If the given angle is measured in degrees, minutes, and seconds or degrees, minutes, and fractions of a minute, the exact value of the desired trigonometric function cannot be read directly from the table. In many instances the value of the function for the angle in the table which is closest to the given angle is used, the next smaller angle being used if the seconds of the given angle is less than 30, and the next larger angle used if the seconds exceed 30.

The reverse process of finding an angle corresponding to a given or computed value of one of the trigonometric functions, when this value of the function lies in between two of those given in the tables, is similarly accomplished. In this case the given function is assumed to be the same as the value of the adjacent value to which it is numerically closest. Where more accurate answers are required than is possible by the procedure described above, the following additional computations are employed.

The operation of finding the value of a variable quantity between those given in a table is called interpolation. One way to interpolate is to plot a smooth graph of the tabulated values and read off the required value corresponding to the given number. Another method, and of more practical importance, is by the use of proportional parts, or by proportion. Using this solution in connection with the trigonometric functions it is necessary to assume that the change in the value of the function (sine, cosine, and tangent, etc.) is directly proportional to the change in the value of the angle. This relationship actually does not exist; however, interpolation carried out in this manner will in general cause no error of any consequence.

For those who are mathematically inclined the following facts in regard to interpolation are included, but to those not interested, it is recommended that the material of this paragraph be omitted and their attention be directed to the paragraph beyond.

The operation of finding the value of any trigonometric function between two adjacent angles given in the table is based on the assumption that the change in the value of the function is directly proportional to the change in the value of the angle. This assumption is not true for the table as a whole inasmuch as it would require, for example, that the sine of 60° should be twice the sine of 30° . As an additional proof of the inaccuracy of the statement, the rate of change of the sine is .00029 for one minute in the small angle range and this rate decreases to approximately zero for angles approaching 90° . The rate of change in the value of the tangent function is even more irregular since it increases at an infinite rate for an angle just less than 90° . However, if it is remembered that the values of the various functions are graduated for each degree and minute or 3600 values for an angular change of 90° , it is apparent that the total change in function for one division of the table is very small. This is especially true for the functions sine and cosine which have a total variation in value of only unity throughout a range of 90° . The value of the tangent function varies from 0 to unity at 45° and above 45° the function increases more rapidly especially with the larger angles which have values of the tangent function approaching infinity as the angle approaches 90° . Since the tabular difference for the tangent function is greater than that for either the sine or cosine function, a small error in the tangent of an angle will affect the angle less than would a corresponding error in either the sine or the cosine. Consequently, an angle should be determined by means of the tangent function wherever practicable.

Also, when the angle is less than 45° , the tabular difference for the sine exceeds that of the cosine, and when the angle is greater than 45° , the tabular difference for the cosine is the greater. Therefore, the angle should be determined by means of its sine rather than its cosine when the angle is less than 45° , and by its cosine rather than its sine when it is greater than 45° . This difference in accuracy among the tangent, sine, and cosine functions is not very great however, and too much emphasis should not be assigned to this choice of functions. Special formulas for precise interpolation have been developed which may be employed when maximum possible accuracy is required.

Interpolation is frequently required in solving problems involving the six trigonometric functions,

sine—cosine—tangent—cotangent—cosecant—secant

The values of the last two of these are not ordinarily included in the tables as they are the reciprocals of the sine and cosine, respectively. The method of finding the value of the tangent is similar to the method of finding the sine of an angle, and the method for finding the cotangent is similar to the method of finding the cosine of an angle. It is therefore necessary to describe only the operations of finding the sine and the cosine of an angle since the same procedure is employed in interpolating for values of the other functions.

Sine of an Angle

The sine of an angle which is given in degrees, minutes and seconds is found by first finding the value of the function corresponding to the major part of the angle as defined in terms of degrees and minutes only. To this value of the function is *added* an additional number resulting from multiplying (*A*), the tabular difference between the sines of the two adjacent angles which appear in the tables, one angle being larger than the given angle, and the other angle being smaller than the given angle, by (*B*), the value of the fraction obtained when the seconds of the given angle are expressed as a fraction of a minute, either as a common fraction as number of seconds/60, or as a decimal.

In many cases the measure of the angle will be given directly in degrees and minutes and decimal parts of a minute thus facilitating the computations being described.

Note that when obtaining the sine of an angle, the value corresponding to the fraction part of a minute or seconds of the angle is always *added* to the value of the function as obtained directly from the table. This is because the value of the sine increases with increasing magnitude of the angle.

18°

/	Sine
sine	
18°	
15.25	
sine 15	.31316
18° 16	.31344

$$\begin{aligned}
 \text{sine } 18^\circ 15.25' &= \\
 \text{sine } 18^\circ 15' &= .31316 \\
 \text{sine } 18^\circ 16' &= .31344 \\
 \text{Tabular difference} &= .00028 \\
 (.25) (.00028) &= .00007 \\
 \text{sine } 18^\circ 15.25' &= .31316 + .00007 \\
 &= .31323
 \end{aligned}$$

Considerable effort can be spared if the tabular difference between the sines of the two adjacent angles which appear in the tables are not treated as decimals. The product of (*A*) and (*B*) is computed without regard to the decimal point of the tabular difference, and the decimal point in the product is then placed by inspection.

SOLUTION OF EQUATIONS

30°	
/	Sine
0	.50000
1	.50025

sine 30° 0 3'

$$\begin{aligned}\sin 30^\circ 0.3' &= \\ \sin 30^\circ 0' &= .50000 \\ \sin 30^\circ 1' &= .50025 \\ \text{Tab Diff.} &= .00025 \\ (3)(25) &= 7.5\end{aligned}$$

If the given angle were included in the tables, it would occupy a position of .3 of the distance between 30° 0' and 30° 1'. The value of its sine would, therefore, be .3 of the total change of (25) units, regardless of the decimal points, the sine of the angle will be (3)(25) or 7.5 units larger than the sine 30° 0'. The proper position to annex the 7.5 is, therefore,

$$\begin{array}{r} .50000 \\ \hline 75 \\ \hline \sin 30^\circ 0 3' = 500075 \\ \sin 30^\circ 0 3' = .50007 \text{ or } 50008 \end{array}$$

The interpolated value can be no more accurate than the table from which it is derived. The number of digits retained in the interpolated value should not be greater than the number of digits occurring in the table being used. The interpolated value in this case is therefore, assumed to be either .50007 or .50008 which are the closest numerical values of five places representing 500075.

The term tabular difference as used above means the difference in the two values of the sine function corresponding to the two angles in the tables between which the given number lies. Note that when interpolating to find the sine of an angle, the product of (A) and (B) is always added to the value of the sine of the major part of the angle. This is because the value of the sine function increases with increasing magnitude of the angle. This statement has been repeated for emphasis.

The process of finding an angle corresponding to a given value of its sine is a reverse process to the procedure just described. However, the solution for such an angle is frequently incorrect, and the following notes and example are included to eliminate any question as to the procedure to be followed.

15°	
/	Sine
40	27001
41	27032

$\sin \theta = 27011$

$$\begin{aligned}\sin \theta &= .27011 \\ \sin 15^\circ 40' &= .27004 \\ \sin 15^\circ 41' &= .27032 \\ \text{Tab. Diff.} &= .00028 \\ \sin 15^\circ 40' &= 27004 \\ \sin \theta &= 27011 \\ \text{Difference} &= .00007\end{aligned}$$

It is apparent that the given angle is larger than 15° 40' by a fraction of a minute which can be expressed as a ratio whose numerator is the amount by which the value of the sine of the given angle exceeds the sine of 15° 40' and whose denominator is the value by which the value of the sine function increases from 15° 40' to 15° 41'.

$$\frac{.00007}{.00028} = \frac{7}{28} = \frac{1}{4} = 25$$

Thus the angle whose sine is .27011 is an angle which is .25 of a minute larger than $15^\circ 40'$. Therefore, $\theta = 15^\circ 40.25'$. The answer obtained is stated in degrees and minutes and fractions of a minute as this is the more convenient form in most cases. If, in some instance, it is necessary to state the measure of the angle in degrees, minutes and seconds, the seconds can be readily computed by the use of the interpolation fraction.

For the same example:

$$\frac{.00007}{.00028} \times 60'' = \frac{7}{28} \times 60'' = \frac{1}{4} \times 60'' = 15''$$

$$\theta = 15^\circ 40' 15''$$

The example as given above,
can also be written

$$\begin{aligned}\sin \theta &= .27011 \\ \sin^{-1} &= .27011\end{aligned}$$

The symbol \sin^{-1} is commonly used to denote "an angle whose sine is ---, whatever value follows. Thus $\sin^{-1} = .5$ or simply $\sin^{-1} .5$ denotes an angle whose sine is .5. A similar notation applies to the use of \cos^{-1} , \tan^{-1} , etc. It should be understood when using this notation that the $^{-1}$ is not an exponent but simply a part of the new symbol for an angle. The symbol \sin^{-1} , \cos^{-1} , etc., is also read arcsine, arccosine, etc. The notation as described is satisfactory to use whenever the given angle or angles in question is not greater than 90° . However, for larger angles some ambiguity results as in the example above, $\sin^{-1} .5$ denotes an angle which might be 30° , 150° , etc. In problems of calculus, where radian measure is regularly used for differentiations and integrations, the meaning of the symbol $\sin^{-1}x$ is restricted to the number of radians in the numerically smallest angle whose sine is (x) . Somewhat similar agreements are made concerning the use of $\cos^{-1}x$, $\tan^{-1}x$, etc.

An exception to the exponential notation described in the first paragraph occurs in the case of the $^{-1}$ power. It is incorrect, for example, to write $(\sin a)^{-1}$ in the abbreviated form $\sin^{-1}a$ since the latter expression denotes an angle whose sine is (a) .

Cosine of an Angle

The cosine of an angle which is given in degrees, minutes, and seconds, is found by first finding the value of the function corresponding to the major part of the angle as defined in terms of degrees and minutes only. To this value of the function is subtracted a number obtained by multiplying (A) the tabular difference between the cosines of the two adjacent angles which appear in the tables, one being larger than the given angle, and the other angle being smaller than the given angle by (B) the value of the fractions obtained when the seconds of the given angle are expressed as a fraction of a minute, either as a formal fraction, as number of seconds/60, or as a decimal.

Note that when interpolating to find the cosine of an angle, the product of (A) and (B) is always subtracted from the value of the cosine of the major part of the angle. This is because the value of the cosine function decreases with increasing magnitude of the angle.

$$\text{Cosine } 40^\circ 40.2' =$$

40°	
/	Cosine
cos 40° 40' 2'	
40	.75851
41	.75832

$$\begin{aligned}\cos 40^\circ 40' &= .75851 \\ \cos 40^\circ 41' &= .75832 \\ \text{Tab Diff.} &= .00019 \\ (2)(.00019) &= .000038 = 0000\frac{1}{4} \\ \cos 40^\circ 40' 2' &= .75851 - .00004 \\ &= .75847\end{aligned}$$

The process of finding an angle corresponding to a given value of its cosine is a reverse process to the procedure just described

$$\cos \theta = 0.86597$$

30°	
/	Cosine
cosine $\theta = 0.86597$	
0	.86603
1	.86588

$$\begin{aligned}\cos 30^\circ 0' &= .86603 \\ \cos 30^\circ 1' &= .86588 \\ \text{Tab Diff.} &= .00015 \\ \cos 30^\circ 0' &= .86603 \\ \cos \theta &= .86597 \\ \text{Diff} &= .00006\end{aligned}$$

It is apparent that the given angle is larger than $30^\circ 0'$ by a fraction of a minute which can be expressed as a ratio whose numerator is the amount by which the value of the cosine of the given angle is less than the cosine of $30^\circ 0'$, and whose denominator is the value by which the value of the cosine function decreases from $30^\circ 0'$ to $30^\circ 1'$.

$$\text{or } \frac{00006}{00015} = \frac{6}{15} = \frac{2}{5} = .4$$

Thus the angle whose cosine is .86597 is an angle which is .4 of a minute larger than $30^\circ 0'$. Therefore $\theta = 30^\circ 0' 4'$.

Although the process of addition is known to be somewhat simpler than the process of subtraction and, therefore, preferable from a mathematical viewpoint, the above described method of interpolation for the cosine function is always recommended instead of an alternate method which would allow addition processes and involve the use of the next higher function in the table than that corresponding to the major part of the angle as defined in terms of degrees and minutes only.

The additional suggestions, rules, and definitions given for the interpolation of the sine function apply equally to the interpolation of the cosine function. In fact the two procedures are the same except that in the case of the cosine function, the product of (A) and (B) is always subtracted from the value of the cosine of the major part of the angle. This is because the value of the cosine function decreases with increasing magnitude of the angle.

Tangent of an Angle

As already stated, the method in interpolation to find the value of the sine of an angle is similar to the method of interpolating to find the value of the tangent of an angle, since the values of both of these functions increase as the size of the angle increases from 0° to 90° .

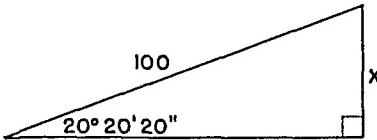
Cotangent, Secant and Cosecant

The trigonometric functions cotangent, secant and cosecant may be called the reciprocal functions since they are the reciprocals of the tangent, cosine and sine respectively. Ordinary tables of the natural trigonometric ratios do not always contain all of the reciprocal functions, but where these are included, interpolation may be performed by the same methods as already described in the preceding paragraphs.

It should be recognized that if a function of an angle increases as the angle increases, then the reciprocal function of the angle will decrease as the angle increases, and vice versa. Consequently, when interpolation is required, observe whether the function being interpolated increases or decreases as the angle increases, and add or subtract as already described for the sine and cosine functions.

SOLUTION OF RIGHT TRIANGLES

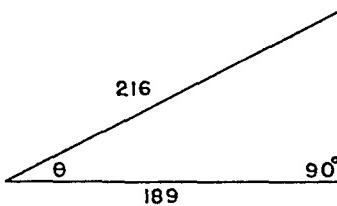
The solution of several right triangles will clearly demonstrate the brevity of the operations as described above.



20°

/	Sine
20	.34748
21	.34775

$$\begin{aligned} \text{in } 20^\circ 20' 20'' &= \frac{x}{100} \\ x &= 100 (\sin 20^\circ 20' 20'') \\ \text{in } 20^\circ 20' 20'' &= \sin 20^\circ 20' \text{ (appr.)} \\ \text{in } 20^\circ 20' &= .34748 \\ x &= (100) (.34748) \\ &= 34.748 \text{ (approx.)} \\ x &= 34.75 \text{ (approx.)} \end{aligned}$$



28°

/	Cosine
57	.87504
58	.87490

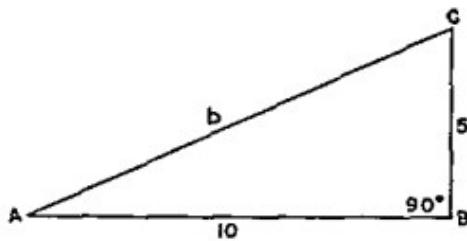
$$\begin{aligned} \cos \theta &= \frac{189}{216} = .87500 \\ \cos 28^\circ 57' &= .87504 = .87500 + .00004 \\ \cos 28^\circ 58' &= .87490 = .87500 - .00010 \\ \theta &= 28^\circ 57' \text{ (approx.)} \end{aligned}$$

The true value of the above two answers are, $x = 34.757$ and $\theta = 28^\circ 57.28'$ respectively.

It is apparent that the results obtained by using the above approximations may be somewhat in error. The magnitude of this error will vary directly as the amount by

which the given value varies from the tabulated values. But even in the extreme cases, the discrepancies are comparatively small and the results obtained are sufficiently accurate for all ordinary purposes. Where more accurate computations are required, the procedure described under interpolation may be employed.

From the foregoing two examples it is apparent that the solution of right triangles for one or more of the unknown elements may be a simple algebraic process once the relationship among the sides has been determined. In many cases, where only one of a possible two or three elements is to be found the most direct solution is at once observed—the most direct solution being also the most logical solution from a labor and accuracy viewpoint. However, where a triangle is to be completely solved for all sides and all angles it is quite possible that a variety of solutions may be employed in finding one or more of these elements. It is the purpose of the following paragraphs to explain the advantages of employing certain procedures in some instances. These suggestions are not to be interpreted as definite rules, however, as the exact method of solution is left to the user's discretion, which will invariably be tempered by the numerical value of the elements concerned.



$$b^2 = 10^2 + 5^2 = 100 + 25 = 125 \quad (\text{Rule 186})$$

$$b = \sqrt{125} = 11.18$$

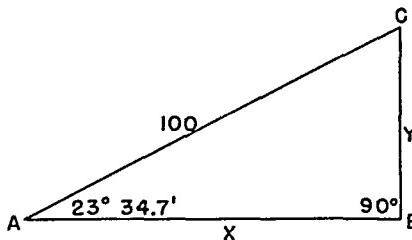
$$\tan A = \frac{5}{10} = .50000$$

$$A = 26^\circ 33.92'$$

$$\tan C = \frac{10}{5} = 2.0000$$

$$C = 63^\circ 26 13'$$

The value of C could have been found by subtracting the value of A from 90° . This is often done in practical work, and is entirely logical provided that the value of A has been precisely computed. If A is in error the value of C will likewise be incorrect. The above solution guards against this possibility and provides a means by which the two answers may be checked for accuracy. If in solving right triangles the value of the second acute angle is not computed independently as suggested above, but is found by subtracting from 90° , then in solving a problem involving the tangent function, the value of the smaller angle should be the one computed if the results obtained are to be as accurate as possible. This difference in accuracy results from erroneous assumption employed in the interpolation process. The magnitude of this error is a maximum for large angle involving the tangent function. For the sine and cosine function this error is of no consequence.



$$\sin 23^\circ 34.7' = \frac{y}{100}$$

$$\begin{aligned} y &= (100) (\sin 23^\circ 34.7') \\ &= (100) (.40000) = 40.00 \end{aligned}$$

$$\cos 23^\circ 34.7' = \frac{x}{100}$$

$$\begin{aligned} x &= (100) (\cos 23^\circ 34.7') \\ &= (100) (.91651) = 91.651 \end{aligned}$$

Check: $C = 90^\circ - 23^\circ 34.7' = 66^\circ 25.3'$
 $100^2 = x^2 + y^2 = 91.65^2 + 40^2$
 $10000 = 8400 + 1600 = 10000$ (Slide Rule)

In this example the value of C is found by subtracting the value of A from 90° . This is the logical procedure inasmuch as the value of A was given and therefore free of any possibility of being in error.

For purposes of demonstrating the value of A was chosen so that its sin would be of such value that when multiplied by the hypotenuse it would produce a rational number which could be conveniently squared. The solution for side (x) is then accomplished by the use of the Pythagorean theorem (Rule 186). This method is sometimes used but is not considered as good practice as the method shown for two reasons, namely:

Only in rare cases will the computed value of (y) be a rational number, or if a rational number, of such magnitude as to be squared by a mental process.

The value of (x) obtained is a computed value and consequently may be in error.

The use of the Pythagorean theorem is, however, recommended as a check on the value of the sides obtained by the method as shown.

In some cases more than one trigonometric function and more than one unknown element is involved in the solution of a given triangle. This requires that more than one equation be written, and that these equations be solved simultaneously.

SOLUTION OF EQUATIONS

$$\begin{array}{rcl}
 (A) (\sin 30^\circ) + (B) (\sin 60^\circ) & = & 100 \\
 (A) (\cos 30^\circ) - (B) (\cos 60^\circ) & = & 0 \\
 \hline
 5A + .866B & = & 100 \\
 866A - 5B & = & 0 \text{ or } .866A = .5B \\
 \hline
 (866)(5A) + (866)(866B) & = & (866)(100) \\
 (5)(866A) - (5)(5B) & = & (5)(0) \\
 \hline
 433A + .75B & = & 86.6 \\
 433A - 25B & = & 0 \\
 \hline
 1.00B & = & 86.6 \\
 B & = & 86.6 \\
 .866A & = & 5B = (.5)(86.6) = 43.3 \\
 A & = & 50
 \end{array}$$

$$(L) (\sin \theta) = \frac{WV^2}{gR}$$

$$(L) (\cos \theta) = W$$

$$\text{Then } \frac{(L) (\sin \theta)}{(L) (\cos \theta)} = \frac{V^2}{gR}$$

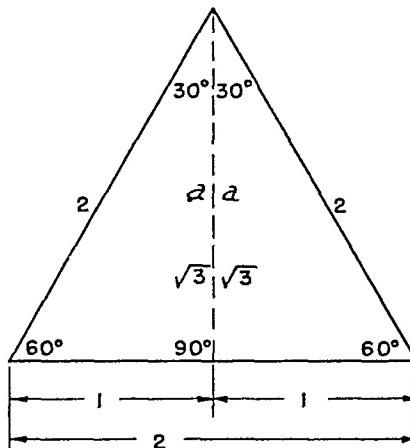
$$\tan \theta = \frac{V^2}{gR}$$

The preceding solutions together with the examples shown on page 137 and 138 are but a few of the many possibilities which are frequently encountered in trigonometric solutions. It is recommended that whenever any solution is attempted, the thought is kept in mind to try to obtain the maximum in accuracy with minimum expenditure of labor. After any solution, the results obtained should be checked for accuracy. For this operation, one or more of the geometric relations appearing on page 130 may be employed. Also, it is possible to construct a triangle from the given data, using a reasonably large scale for the drawing. The value of the unknown elements may be determined from this figure using a protractor and rule, and the results obtained compared to those obtained by the algebraic computation. This graphical check is only approximate, but is very useful in such instances where such a degree of accuracy is permissible.

Special Solution of $30^\circ\text{-}60^\circ$ and 45° Right Triangles

Right triangles containing one 30° and one 60° angle (called a $30^\circ\text{-}60^\circ$ right triangle) are so frequently encountered that it is advisable to remember the values of the \sin , \cos and \tan for these two angles. Right triangles containing two 45° angles are also of common occurrence for which the common trigonometric functions are often required. The various functions for these three angles can easily be memorized by employing a simple fractional notation as now derived.

Ratios of 30° and 60°



The trigonometric functions of 30° and 60° can be found by the use of an equilateral triangle. This may be of any arbitrary size but to simplify the computations which are to follow the dimension of each side is assumed to be *two* units in length in any system of measurement. An altitude drawn to any vertex bisects that angle forming two 30° angles. The side opposite the vertex is also bisected forming two equal segments of unit length. The altitude therefore divides the given equilateral triangle into two identical right triangles which have a hypotenuse two units in length and the shorter side of one unit in length. By the Pythagorean theorem, the length of the remaining side is found to be $\sqrt{3}$ or 1.732 since,

$$2^2 = 1^2 + a^2$$

$$\begin{aligned} a^2 &= 4 - 1 = 3 \\ a &= \sqrt{3} = 1.732 \dots \end{aligned}$$

The trigonometric functions of 30° and 60° can now be derived.

$$\sin 30^\circ = \frac{1}{2} = .500$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866 \dots$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{1}{1.732} = .577 \dots$$

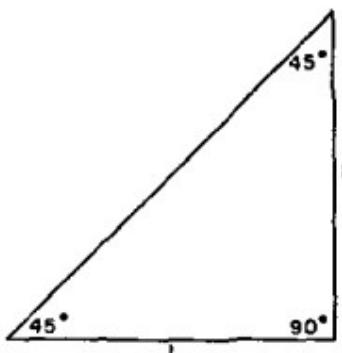
$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866 \dots$$

$$\cos 60^\circ = \frac{1}{2} = .500$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1} = \frac{1.732}{1} = 1.732 \dots$$

Ratios for 45°

$$\begin{aligned} b &= \\ \sqrt{2} &\approx 1.414 \end{aligned}$$



The trigonometric functions for 45° can be found by the use of an isosceles triangle. This may be of any arbitrary size, but to simplify the computations which are to follow the dimension of the two equal-length sides is assumed to be *one unit* in length in any system of measurement. By the Pythagorean theorem, the length of the hypotenuse is found to be $\sqrt{2}$ or 1.414 (since $b^2 = 1^2 + 1^2 = 2$ Therefore $b = \sqrt{2}$)

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{1.414\dots} = .707\dots$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{1.414\dots} = .707\dots$$

$$\tan 45^\circ = \frac{1}{1} = 1$$

From the foregoing derivations,

$$\sin 30^\circ = \frac{1}{2}$$

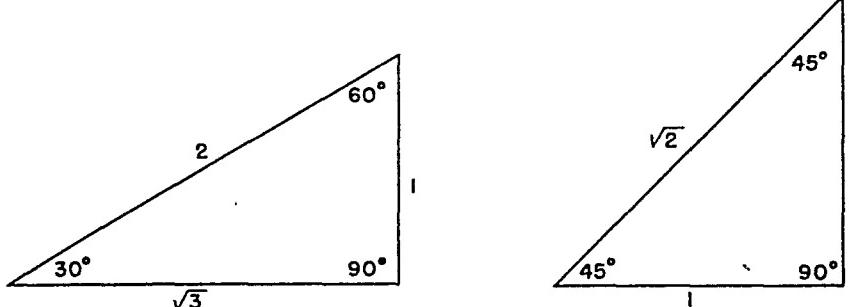
$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

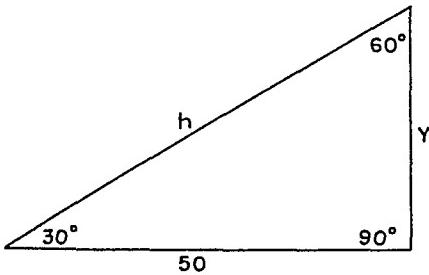
The sine of 30° can also be written as $\sqrt{1}/2$, and the sine of 45° can be written as $\sqrt{2}/2$ if both the numerator and the denominator of the original fraction are multiplied by the order named as $\sqrt{1}/2$, $\sqrt{2}/2$ and $\sqrt{3}/2$.

The previously described methods of deriving the sin, cos and tan for 30°, 45° and 60° makes it possible to solve triangles in which these angles occur without the use of a table of the trigonometric functions.

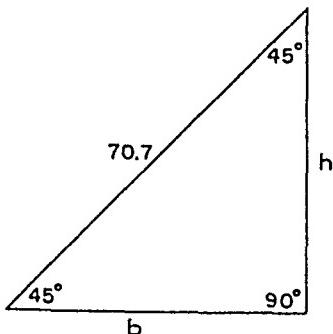
A further simplification in solving 30°-60° and 45° right triangles is possible by employing the theorem of geometry that states that the corresponding sides of similar triangles are proportional. An inspection of the diagrams on page 141 and above shows that the relative lengths of the sides of these triangles can be represented by either small integers or their square roots. These values are shown below.



Whenever any given triangle is known to be similar to one of the above types, the length of its sides may be determined, provided that the length of one of the sides is known. If the triangles are similar with only the angles given, there can be an infinite number of triangles of the same shape but of varying areas which will satisfy the stated conditions. Consequently, the values of the sides cannot be found.



$$\begin{aligned} \frac{b}{50} &= \frac{2}{\sqrt{3}} \\ b &= \frac{(50)(2)}{\sqrt{3}} = \frac{100}{1.732} \\ &= 57.8 \text{ (approx.)} \end{aligned}$$



$$\begin{aligned} \frac{70.7}{b} &= \frac{\sqrt{2}}{1} \\ b &= \frac{70.7}{\sqrt{2}} = \frac{70.7}{1.414} = 50 \\ b &= b = 50 \end{aligned}$$

Reciprocal Functions

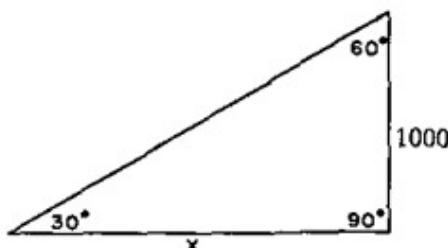
The trigonometric functions cotangent, secant and cosecant may be called the reciprocal functions since they are the reciprocals of the tangent, cosine and sine, respectively. The reciprocal functions are not necessary for the solution of right-triangles, but their use in some cases may simplify the computations involved. Thus, the result obtained by dividing some side by the sine, cosine or tangent of an angle can be more easily found by multiplying that side by the reciprocal of the functions involved. This fact is of importance only when such computations are performed long-hand and is of no consequence when a slide-rule or calculating machine is employed.

Ordinary tables of the natural trigonometric functions do not always include all of the reciprocal functions, but in most cases the cotangent, at least, is represented. This function is employed in the example below in which an operation of division is re-

SOLUTION OF EQUATIONS

placed by one of multiplication. Similar applications are possible with the other reciprocal functions shown by the solution of the two additional examples on the following page.

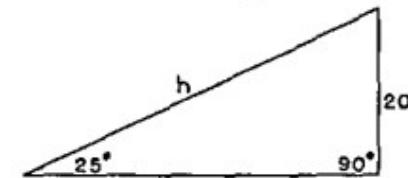
Example 1.



$$\tan 30^\circ = \frac{1000}{x}$$

$$x = \frac{1000}{\tan 30^\circ} = (1000) \left(\frac{1}{\tan 30^\circ} \right) = 1000 \cot 30^\circ \\ = (1000) (1.732051) = 1732.051$$

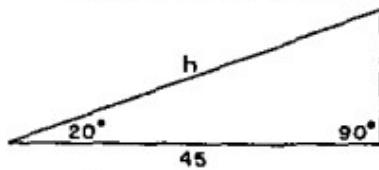
Example 2



$$\sin 25^\circ = \frac{20}{b}$$

$$b = \frac{20}{\sin 25^\circ} = (20) \left(\frac{1}{\sin 25^\circ} \right) = (20) (\cosec 25^\circ) \\ = (20) (2.3662) = 47.324$$

Example 3.



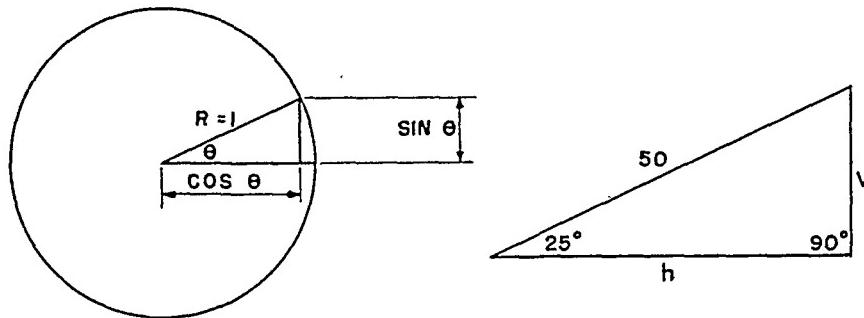
$$\cos 20^\circ = \frac{45}{b}$$

$$b = \frac{45}{\cos 20^\circ} = (45) \left(\frac{1}{\cos 20^\circ} \right) = (45) (\sec 20^\circ) \\ = (45) (1.1034) = 49.653$$

Special Solution of Right Triangles Having the Hypotenuse Given

The frequent occurrence in the study of mechanics of the need for finding the vertical and horizontal components of a given force makes it desirable to know a method of solution with the least amount of effort involved. Such a problem is equivalent to finding the other two sides of a right triangle when the hypotenuse and one or both of

the acute angles are given. If one acute angle is given the value of the second acute angle can be obtained by a simple subtraction. Therefore, having one acute angle given is practically the same as though both were given.



An inspection of the diagram on the left indicates that in a circle with a radius of unity the numerical values of the trigonometric ratios of sine and cosine are actually equal to the length of the lines as shown. This fact is established on page 148. The value of the sine and cosine for *any angle whatsoever* is always a decimal fraction as these functions are represented by the sides of a right triangle which has a hypotenuse (radius) equal to unity.

A comparison of the above two figures shows them to be directly comparable, each having the same kind and number of elements given. The hypotenuse of the right triangle in the diagram on the left is 1, while that of the right triangle on the right is equal to 50. If, for an example, θ is arbitrarily assumed to be 25° , then $\sin \theta$ will have a numerical value corresponding to $\sin 25^\circ$, or .42262. Similarly, $\cos \theta$ will have numerical value corresponding to $\cos 25^\circ$ or .90631. With θ assumed to be 25° the two right-triangles are similar.

$$\text{Therefore, } \frac{50}{1} = \frac{v}{.42262}$$

$$v = (50) (.42262) = 21.131$$

$$\frac{50}{1} = \frac{b}{.90631}$$

$$b = (50) (.906308) = 45.3154$$

The equations for finding (v) and (b) can be written directly without first establishing the proportions as shown above if the idea of the unit-radius circle is kept in mind.

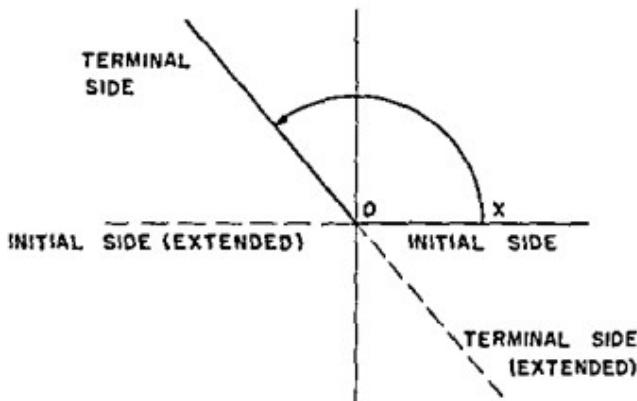
TRIGONOMETRIC FUNCTIONS IN OBLIQUE TRIANGLES

The previous definitions of the trigonometric functions, as the ratios of the lengths of the various sides in a right triangle, becomes confusing when oblique triangles are being solved by the use of special oblique triangle formulas. For this reason an alternate explanation of the functions is presented, in which the definitions are not associated with right triangles, but with any angle. A knowledge of polar coordinates is essential for this explanation.

Polar Coordinates

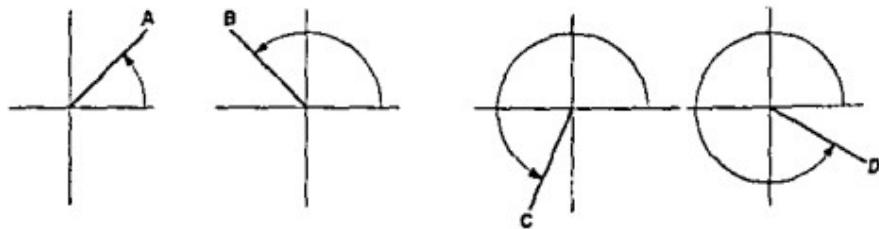
It has already been shown on page 39, that any point in a place can be located by means of Cartesian coordinates.

A second method of locating a point in a plane makes use of angular measurement together with the distance of the point from the *pole*. The pole in the polar coordinate system is the same as the origin in the Cartesian system. In mathematical work, angular measurement commences on the positive side of the X - X axis and increases positively with counter-clockwise rotation about the pole. An angle is therefore bounded by the positive side of the X - X axis, called the *initial side*, and by the *terminal side* which corresponds to a radius which has been rotated about the pole to produce an angle of the desired magnitude



The distance of the point from the pole is determined its *radius vector* or simply *radius*, and is further abbreviated as (r). The radius is always considered a positive quantity when laid off on the terminal side of the angle, and negative if laid off on the terminal side produced (extended) though the pole.

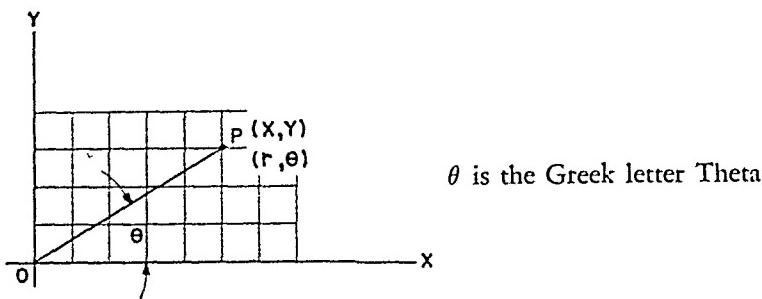
In designating a point, the radius vector is stated first, then the angle. Thus $A(5, 45^\circ)$, $B(8, 135^\circ)$, $C(12, 250^\circ)$, and $D(15, 330^\circ)$ represent points in the first, second, third and fourth quadrants, respectively.



To plot a point whose polar coordinates are given, begin by drawing a line on which the radius vector lies—that is a line of indefinite length, but making the specified angle with the initial side. Next, lay off on that line the radius vector using a convenient scale of distances.

Transformation from Polar to Cartesian Coordinates or Vice Versa

If the pole and the initial side of a polar system coincide with the origin and the OX axis of a Cartesian system, the coordinates of any point in the two systems are readily convertible.

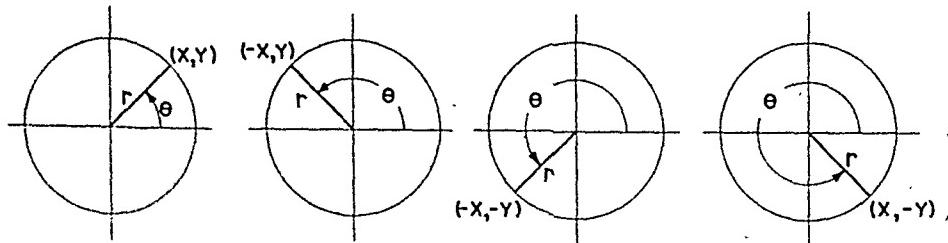


Let the point P have the Cartesian coordinates (x, y) and the polar coordinates (r, θ) . Using the trigonometric relations, we have:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ \text{also, } & r = \sqrt{x^2 + y^2} \\ & \tan \theta = y/x \end{aligned}$$

These equations enable the transformation of the coordinates of a point from one system to the other whenever such transformation is desired.

By the use of polar coordinates as described above, it is possible to define the trigonometric functions for any angle regardless of its magnitude, that is, whether acute or obtuse.



$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

$$\operatorname{cosec} \theta = \frac{r}{y}$$

$$\sec \theta = \frac{r}{x}$$

$$\cotan \theta = \frac{x}{y}$$

The algebraic sign (+) or (-) to be assigned to any particular functions depends upon the quadrant in which the angle lies. For example, the values of both (x) and (y) are positive in the first quadrant, and since the radius vector, (r) , is always positive, the values of all six trigonometric functions in the first quadrant are positive.

The value of any function in any other quadrant is determined by the sign of (x) and (y) in that quadrant. Thus, the cosine of any angle lying in the second quadrant is negative, since the value of (x) is negative.

GEOMETRIC REPRESENTATION OF THE TRIGONOMETRIC FUNCTIONS

The trigonometric functions can also be represented by the lengths of certain lines in a circle. If the *radius* of the circle is taken as *unity* (one unit in any system) then the trigonometric ratios are actually equal to the length of these lines measured to the same scale as the radius.

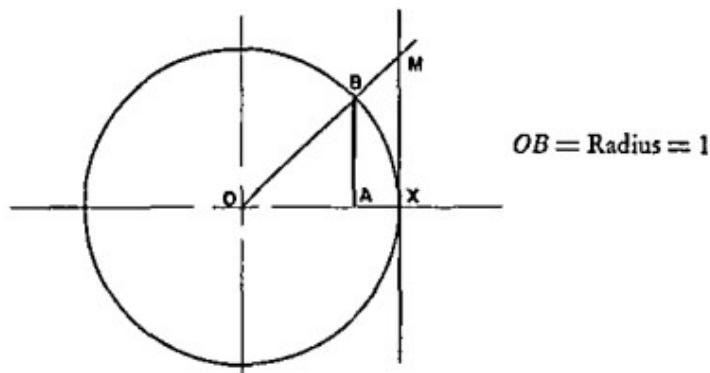


Figure 1

With a radius of *unity* and the vertex of the angle at the center, a circle can be described and a tangent drawn at the point (X) where the initial side (OA) cuts the circle. The trigonometric functions of sin, cos and tan, are as follows:

$$\begin{aligned} \sin \theta &= AB/OB \\ \text{but, } & \quad OB = 1 \\ \sin \theta &= AB/1 = AB \end{aligned} \qquad \qquad \qquad (\text{Fig. 1})$$

Definition: The sin of an angle is the perpendicular distance from the point where the terminal side of the angle cuts the circle, to the initial side OA (extended if necessary as for angles in the range 90° to 270°).

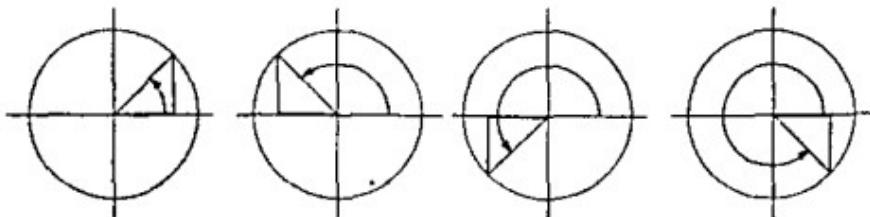


Figure 2

The sin is positive (+) when measured above the X-X axis, and negative (-)

when measured below the X-X axis. The sin is therefore always positive when the angle is in the first or second quadrant, and negative in the third or fourth quadrant.

$$\cos \theta = OA/OB = OA/1 = OA$$

(Fig. 1)

Definition: The cosine of an angle is the projection of the terminal side on the X-X axis, or as might be stated, the distance from the center, (origin or pole), to the point where the perpendicular line representing the sin cuts the initial side (extended if necessary, as for angles in the range 90° to 270°).

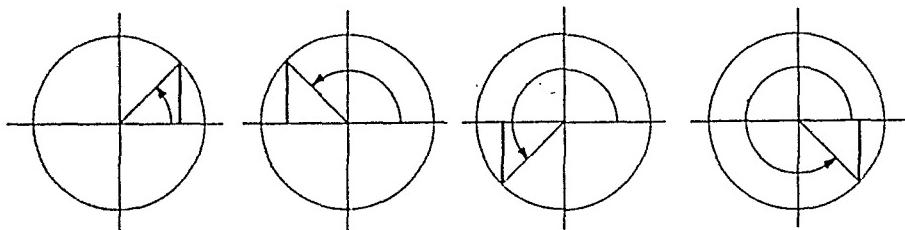


Figure 3

The cosine is positive (+) when measured to the right of the Y-Y axis, and negative (-) when measured to the left of the Y-Y axis. The cosine is therefore always positive when the angle is in the first or fourth quadrants, and negative when in the second or third quadrants.

Definition: The tangent of an angle is the distance along the line tangent to the circle, from the point where the initial side cuts the circle to the point where the terminal side of the angle cuts the tangent line. (Extend the terminal side of the angle if necessary, as for angles in the range 90° to 270°).

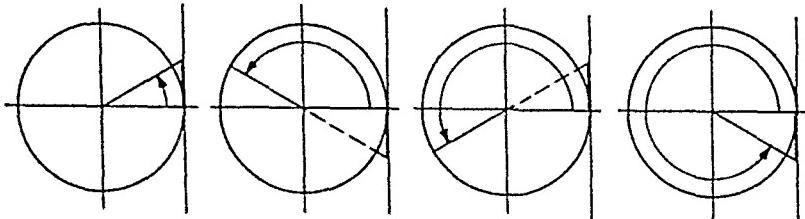


Figure 4

The tangent of an angle is positive (+) when measured above the X-X axis, and negative (-) when measured below the X-X axis. The tangent of an angle is therefore always positive when the angle is in the first or third quadrant, and negative in the second or fourth quadrant. It may be easier for some to remember that the $\tan \theta = \sin \theta / \cos \theta$.

$\theta/\cos \theta$, (Rule 224) and from this relationship, the prefix sign for the tan of any angle can be readily determined by substituting the signs for the sin and the cos for this same angle.

VALUE OF THE FUNCTIONS OF OBTUSE ANGLES

It is sometimes necessary to find the value of one or more of the trigonometric functions for angles greater than 90° . The ordinary tables of the natural trigonometric functions include only those angles from 0° to 90° . However, it can be shown that the sin, cos, tan, etc., of any angle over 90° is the same in magnitude as some acute angle which will be included in the tables. The rule for finding this equivalent acute angle is as follows:

Take the difference between the given obtuse angle and 180° or 360° , whichever gives an acute angle, and prefix the proper sign (\pm) to the value of the function according to the quadrant in which the original (obtuse) angle lies.
(Rule 214)

If an angle (such as 225°) after being subtracted from 360° does not yield an acute angle, then a second subtraction is performed to find the difference between 180° and the difference as already found between 360° and the given angle.

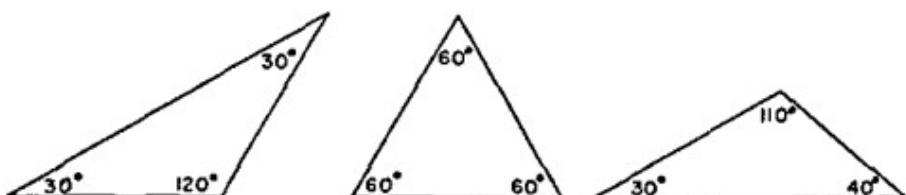
$$\begin{aligned}360^\circ - 225^\circ &= 135^\circ \\180^\circ - 135^\circ &= 45^\circ\end{aligned}$$

The validity of the rule as stated above should be apparent from an examination of figures 2, 3 and 4. It should also be evident that any particular trigonometric function of two angles will be equal in magnitude if one or more of the following conditions are fulfilled:

1. That the angles differ by 180° .
2. That the angles differ by equal amounts from the X-X axis (from 0° or 180°).
3. That the angles differ by equal amounts from the Y-Y axis (from 90° to 270°).

OBLIQUE TRIANGLES SOLVED AS RIGHT TRIANGLES

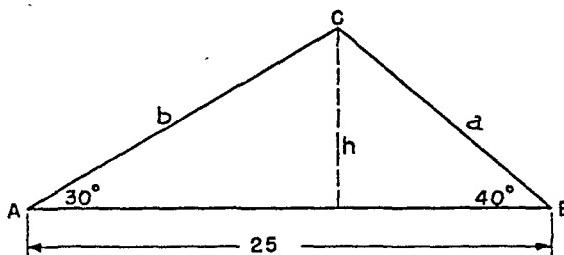
Oblique triangles are triangles which do not contain a right triangle (90°). Such triangles may or may not contain one angle greater than a right angle as shown below.



The solution of oblique triangles, in general, is most easily accomplished by the use of special formulas of which the cosine law and the sine proportion are the most commonly used. These are demonstrated by examples in later paragraphs. However, there

are many instances in which oblique triangles can be solved by the methods previously described for the solution of right triangles. A few of these are included for the purpose of demonstration. When solving oblique triangles as right triangles it is usually possible to employ any one of several different procedures.

Two Angles and Any Side. (This is equivalent to having all three angles and any side given.)



Draw CD or (b) perpendicular to AB forming right-triangles ACD and CDB .

$$b = b \sin 30^\circ$$

$$b = a \sin 40^\circ$$

$$b \sin 30^\circ = a \sin 40^\circ$$

$$b \sin 30^\circ - a \sin 40^\circ = 0$$

$$\underline{b \cos 30^\circ + a \cos 40^\circ = 25}$$

$$\begin{array}{r} .5 b \\ -.64279 a = 0 \\ \hline .86603 b + .76604 a = 25 \end{array} \quad \text{or } b = 1.28558 a$$

$$(.86603)(1.28558 a) + .76604 a = 25$$

$$1.11335 a + .76604 a = 25$$

$$1.87939 a = 25$$

$$1.88 a = 25$$

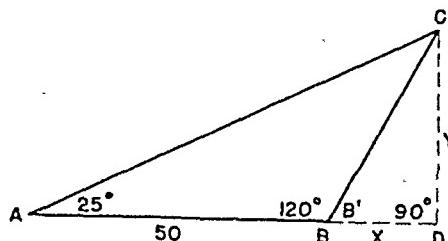
$$a = 13.297 = 13.30 \text{ (approx.)}$$

$$b = (1.28558)(13.30)$$

$$= 17.10 \text{ (approx.)}$$

$$C = 180^\circ - (30^\circ + 40^\circ) = 110^\circ$$

Two Angles and Any Side. (This is equivalent to having all three angles and any side given.)



Extend AB to D forming right-triangle ACD . Denote BD by (x) and CD by (y).

SOLUTION OF EQUATIONS

$$B' = 180^\circ - 120^\circ = 60^\circ$$

$$\tan 25^\circ = \frac{y}{50+x} \quad \text{or} \quad y = .46631(50+x)$$

$$\tan 60^\circ = \frac{y}{x} \quad \text{or} \quad y = 1.7320x$$

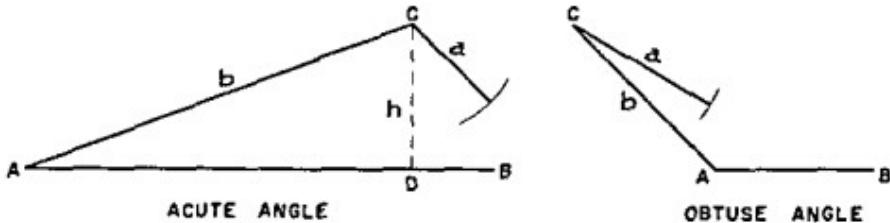
$$\begin{aligned} 1.7320x &= .46631(50+x) = 23.3155 + .46631x \\ 1.7320x - .46631x &= 23.3155 \\ 1.26569x &= 23.3155 \\ x &= 18.42 \text{ (approx.)} \\ y &= (1.7320)(18.42) \\ &= 31.90 \text{ (approx.)} \end{aligned}$$

$$BC = \frac{31.90}{\sin 60^\circ} = \frac{31.90}{.86603} = 36.836 = 36.84\dots$$

$$AC = \frac{31.90}{\sin 25^\circ} = \frac{31.90}{.42262} = 75.48\dots$$

$$\begin{aligned} C &= 180^\circ - (25^\circ + 120^\circ) \\ &= 180^\circ - 145^\circ \\ &= 35^\circ \end{aligned}$$

Two sides and an angle opposite one of them—Ambiguous Case. An oblique triangle which has only two sides and an angle opposite one of them given is known as the ambiguous case, inasmuch as there may be no solution, one solution, or two solutions.



In the oblique triangle shown above, assume that the two sides and the angle which are given are the sides (a) and (b), and the angle A which is opposite side (a). Angle A may be either an acute or an obtuse angle.

To get any idea of the different types of triangles which may be constructed from the given elements, it is convenient to think of any two of the given elements as fixed, and vary only the magnitude of the third.

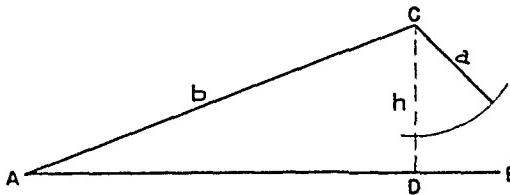
Consequently, it is assumed that angle A and side (b) remain unchanged. For the present discussion, angle A is considered an acute angle.

A triangle is constructed from the given data by first drawing side (b) and angle A . Side (b) is also lettered AC . At end C an arc is swung using the length of side (a) as a radius. In order for side (a) to close the triangle, this side should be long enough to reach the opposite side at some point, B . By drawing CD perpendicular to side AB , a right triangle, ADC is formed. The length of CD , or (b), is:

$$CD = b = b \sin A$$

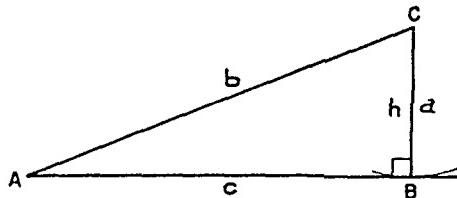
An examination of the figure above shows that the following possibilities may occur:

1.



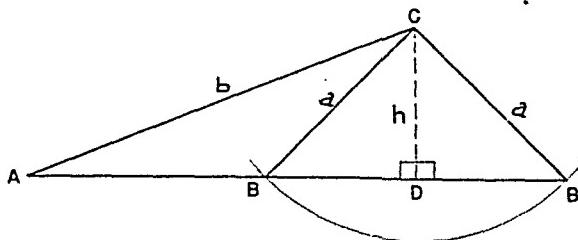
If side (a) is less than (b), it will not be long enough to reach the line ADB , in which case there can be no triangle.

2.



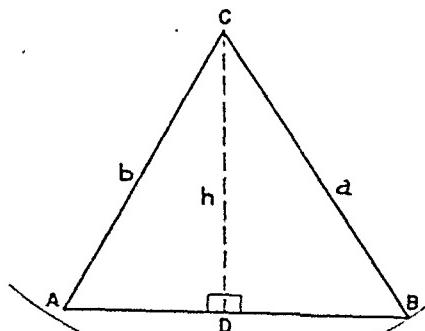
If side (a) equals side (b), then (a) coincides with (b), and the triangle is a right triangle with the right angle at B .

3.



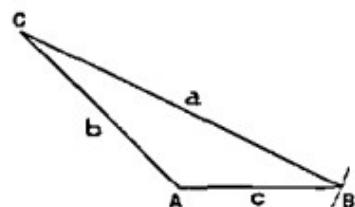
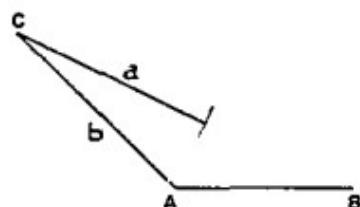
If side (a) is greater than (b), but less than (b), there will be two oblique triangles, ABC and $AB'C$.

4.



If side (a) is either greater than or equal to (b), it will be greater than (b), and there is only one triangle, ABC .

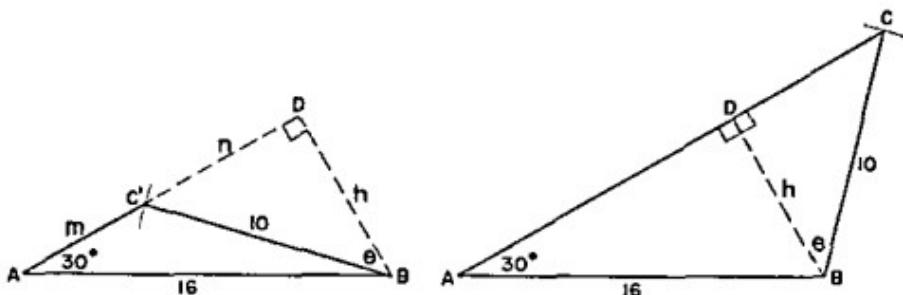
Now if angle A is obtuse, and side (b) remains unchanged as before, the following additional possibilities may occur:



5. If side (a) is either less than or equal to (b), there will be no triangle.
6. If side (a) is greater than (b), there will be only one triangle.

It is apparent that if two sides and an angle opposite one of them is given, there may be no solution, one solution, or two solutions, all depending on the relative magnitudes of the given elements. In order that there may be two solutions, the given angle must be acute, and the side opposite it must be less than the side adjacent.

Two Sides and an Angle Opposite One of Them—Ambiguous Case.



Draw BD or (h) perpendicular to AC' and AC forming right triangles ADB .

In triangle ABC'

$$b = 16 \sin 30^\circ = 8$$

$$(m+n)^2 = 16^2 - 8^2 = 256 - 64 = 192$$

$$AD = (m+n) = \sqrt{192} = \pm 13.86$$

$$CD = n = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$

$$AC' = m = 13.86 - 6 = 7.86$$

$$\cos \theta = \frac{8}{10} = .80000$$

$$\theta = 36^\circ 52.18'$$

$$B = 90^\circ - 30^\circ - 36^\circ 52.18'$$

$$B = 23^\circ 7.82'$$

$$C = 180^\circ - 30^\circ - 23^\circ 7.82'$$

$$C = 126^\circ 52.18'$$

In triangle ABC

$$AC = AD + 6 = 13.86 + 6 = 19.86$$

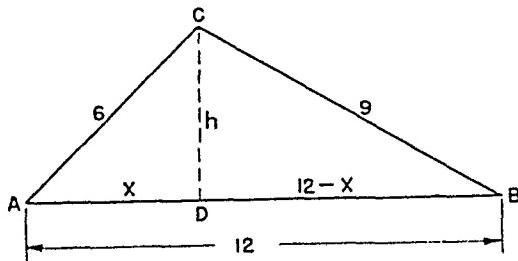
$$B = 90^\circ - 30^\circ + 36^\circ 52.18'$$

$$B = 96^\circ 52.18'$$

$$C = 180^\circ - 30^\circ - 96^\circ 52.18'$$

$$C = 53^\circ 7.82'$$

Three sides given:



Draw CD or (b) perpendicular to AB forming right-triangles ACD and CDB .

$$6^2 = b^2 + x^2$$

$$b^2 = 36 - x^2$$

$$9^2 = b^2 + (12 - x)^2$$

$$b^2 = 81 - (12 - x)^2$$

$$36 - x^2 = 81 - 144 + 24x - x^2$$

$$99 = 24x$$

$$x = 4.125$$

$$12 - x = 12 - 4.125 = 7.875$$

$$\cos A = \frac{4.125}{6} = .68750$$

$$A = 46^\circ 34.05'$$

$$\cos B = \frac{7.875}{9} = .87500$$

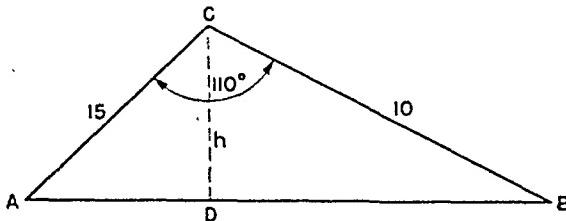
$$B = 28^\circ 57.29'$$

$$= 180^\circ - (A + B)$$

$$= 180^\circ - (46^\circ 34.05' + 28^\circ 57.29')$$

$$= 180^\circ - 75^\circ 31.34' = 104^\circ 28.66'$$

Two Sides and the Included Angle



Draw CD or (b) perpendicular to AB forming right-triangles ACD and CDB .

$$A + B = 180^\circ - 110^\circ = 70^\circ$$

$$A = 70^\circ - B$$

$$\therefore (15) (\sin A) = b \quad (10) (\sin B) = b$$

$$(15) (\sin A) = (10) (\sin B)$$

$$15 \sin (70^\circ - B) = 10 \sin B$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \quad (\text{Rule 231})$$

$$15 (\sin 70^\circ \cos B - \cos 70^\circ \sin B) = 10 \sin B$$

$$15 (.93969 \cos B - .34202 \sin B) = 10 \sin B$$

$$14.09535 \cos B - 5.13030 \sin B = 10 \sin B$$

$$14.09535 \cos B = 10 \sin B + 5.13030 \sin B$$

$$14.09535 \cos B = 15.13030 \sin B$$

$$\frac{14.09535 \cos B}{15.13030 \cos B} = \frac{15.13030 \sin B}{15.13030 \cos B}$$

$$0.93160 = \tan B$$

$$B = 42^\circ 58.31'$$

$$A = 180^\circ - (110^\circ + 42^\circ 58.31')$$

$$= 180^\circ - 152^\circ 58.31'$$

$$= 27^\circ 1.69'$$

$$AB = AD + DB$$

$$= (15) (\cos 27^\circ 1.69') +$$

$$(10) (\cos 42^\circ 58.31')$$

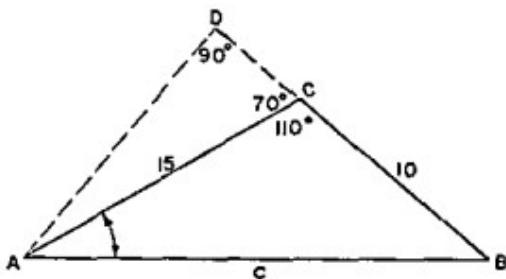
$$= (15) (.89078) + (10) (.73169)$$

$$= 13.3619 + 7.3169 = 20.6788$$

$$= 20.68 \text{ (approx.)}$$

Two Sides and the Included Angle

Alternate Solution



Extend CB to D forming right-triangle ABD

$$BD = (15) (\cos 70^\circ) + 10 = (15) (.34202) + 10$$

$$= 5.13030 + 10 = 15.13030$$

$$= 15.13 \text{ (approx.)}$$

$$AD = (15) (\sin 70^\circ) = (15) (.93969) = 14.09535$$

$$= 14.10 \text{ (approx.)}$$

$$\tan B = \frac{14.09535}{15.13030} = .93160$$

$$B = 42^\circ 58.31'$$

$$A = 180^\circ - (42^\circ 58.31') + 110^\circ = 180^\circ - 152^\circ 58.31'$$

$$= 27^\circ 1.69'$$

$$c^2 = (BD)^2 + (AD)^2 = (15.13)^2 + (14.10)^2 = 228.92 + 198.81 = 427.73$$

$$c = \sqrt{427.73} = 20.68 \text{ (approx.)}$$

OBLIQUE TRIANGLES SOLVED BY SPECIAL FORMULAS

In many cases oblique triangles can be solved by the same methods used in the solution of right triangles. Such a procedure has been described and a variety of examples have been solved to show the simplicity of the method in some cases and the complexity of the solution in other instances. Another conclusion which should be drawn from these examples is that it is often necessary to use computed values in succeeding computations, a policy to be avoided whenever possible.

The solutions of oblique triangles by the use of special formulas seems to be more logical than the searching for relationships which will permit the solution of the problem as a series of right triangles. Furthermore, the use of these formulas is a more direct solution in most cases, and the necessity of using computed values is less frequently required, if at all.

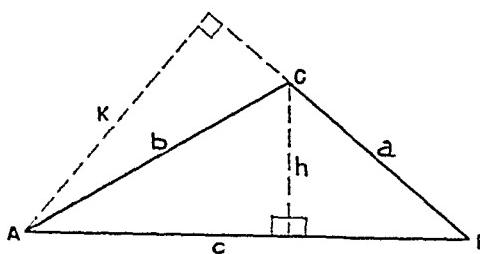
Sine Proportion

Of several formulas applying to oblique triangles, only the sine portion and the cosine law will be described for solutions in which computations are to be made using the slide rule, long-hand, or calculating machine. For the solution of oblique triangles using logarithms, the cosine law is not a practical solution. For this reason, an additional formula adapted to logarithmic computations is stated and proved. This additional method is known as the tangent law.

The sine proportion is used to solve oblique triangles when only the following combinations of angles and sides are known:

1. Two angles and any side. (This is the equivalent of having all the angles and any side given).
2. Two sides and an angle opposite one of them.

The derivation of the sine proportion is as follows:



$$b = b \sin A$$

$$b = a \sin B$$

$$b \sin A = a \sin B$$

$$k = c \sin B$$

$$k = b \sin C$$

$$c \sin B = b \sin C$$

$$\text{or, } \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\text{or, } \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Where (*a*) is the side opposite angle *A*, (*b*) is the side opposite angle *B*, and (*c*) is the side opposite angle *C*.

The sine proportion may be stated.

The ratio of the length of any side to the sine of the opposite angle is a constant. (Rule 215)

also,

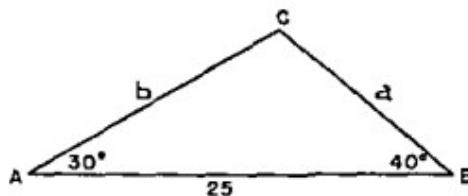
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

It is evident that if any three of the elements in one of these proportions are given, the fourth element can be found. Then, using the value of the element thus found in another proportion, the remaining element can be found, completing the solution.

Two angles and any side. (This is equivalent to three angles and any side).



$$C = 180^\circ - (30^\circ + 40^\circ) = 180^\circ - 70^\circ = 110^\circ$$

$$\sin 110^\circ = \sin (180^\circ - 110^\circ) = \sin 70^\circ = .939693$$

$$\frac{a}{\sin 30^\circ} = \frac{25}{\sin 110^\circ}$$

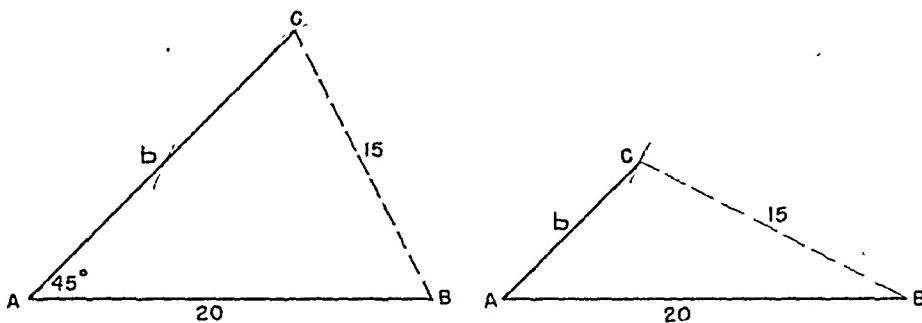
$$a = \frac{(5)(25)}{.93969} = \frac{12.5}{.93969} = 13.30 \text{ (approx.)}$$

$$\frac{b}{\sin 40^\circ} = \frac{25}{\sin 110^\circ}$$

$$b = \frac{(.64279)(25)}{.93969} = 17.10 \text{ (approx.)}$$

Two sides and an angle opposite one of them.—Ambiguous Case

An oblique triangle which has only two sides and an angle opposite one of them given is known as the ambiguous case inasmuch as there may be either no solution, one solution or two solutions. These possibilities are explained on page 152. Whether there is no solution, one solution, or two solutions can always be determined by making a careful construction of the triangle from the given elements.



$$\frac{15}{\sin 45^\circ} = \frac{20}{\sin C}$$

$$\frac{15}{.70711} = \frac{20}{\sin C}$$

$$15 \sin C = 20 \times .70711 = 14.1422$$

$$\sin C = \frac{14.1422}{15} = .94281$$

$$C = 70^\circ 31.78'$$

Since the value of angle C is determined by means of its sine, it may have two values, one angle being less than 90° and the other angle greater than 90° , and their sum being 180° . Therefore, there may be two triangles drawn from the given elements. In one triangle, angle C is $70^\circ 31.78'$ as determined above, and in the other triangle, angle C is $180^\circ - 70^\circ 31.78'$ or $109^\circ 28.22'$.

The value of angle B will likewise vary depending on whether angle C is $70^\circ 31.78'$ or $109^\circ 28.22'$. Consequently, the value of side (b) will also have two values, since side (b) is determined by the sin proportion in which the value of angle B is used.

$$B = 180^\circ - 45^\circ - 70^\circ 31.78' \\ = 64^\circ 28.22'$$

$$B = 180^\circ - 45^\circ - 109^\circ 28.22' \\ = 25^\circ 31.78'$$

$$\frac{b}{\sin 64^\circ 28.22'} = \frac{15}{\sin 45^\circ}$$

$$\frac{b}{\sin 25^\circ 31.78'} = \frac{15}{\sin 45^\circ}$$

$$\frac{b}{.90236} = \frac{15}{.70711}$$

$$\frac{b}{.43098} = \frac{15}{.70711}$$

$$.70711b = 13.5354 \\ b = 19.14 \text{ (approx.)}$$

$$.70711b = 6.4647 \\ b = 9.14 \text{ (approx.)}$$

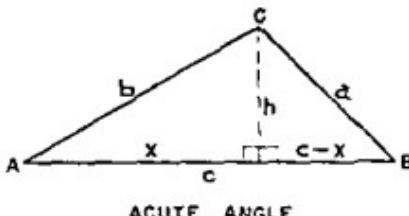
Cosine Law

By referring to the sine proportion, it is evident that the sine proportion may be employed whenever a sufficient number and kind of elements are given so that a proportion involving only one unknown can be established between any two of the ratios. This requirement precludes the use of the sine proportion when either of the following combination of elements are all that are given.

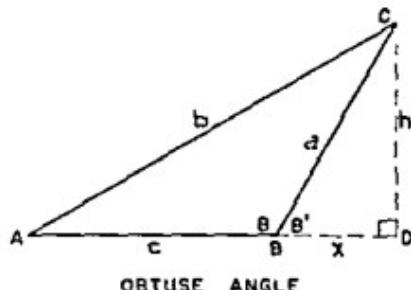
1. Three sides

2. Two sides and the included angle.

The derivation of the cosine law is as follows.



ACUTE ANGLE



OBTUSE ANGLE

$$\begin{aligned} h^2 &= b^2 - x^2, \text{ and } b^2 = a^2 - (c - x)^2 \\ &\quad = a^2 - c^2 + 2cx - x^2 \\ b^2 - x^2 &= a^2 - c^2 + 2cx - x^2 \\ a^2 &= b^2 + c^2 - 2cx \end{aligned}$$

$$\cos A = \frac{x}{b}, \text{ or } x = b \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

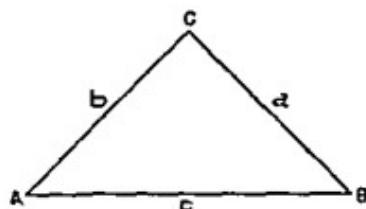
$$\begin{aligned} b &= b \sin A \\ x &= b \cos A - c \\ a^2 &= b^2 + x^2 = (b \sin A)^2 + \\ &\quad (b \cos A - c)^2 \end{aligned}$$

$$\begin{aligned} &= b^2 \sin^2 A + b^2 \cos^2 A - 2bc \cos A + c^2 \\ &= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A \\ \text{but, } \sin^2 A + \cos^2 A &= 1 \\ a^2 &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

The cosine law may be stated

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of the two sides multiplied by the cosine of their included angle.

(Rule 216)



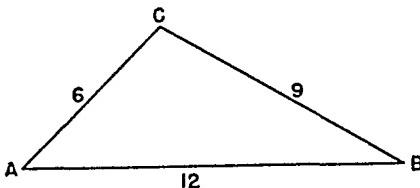
Similarly,
Similarly,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

It is not recommended that the equations as written be memorized because the letters assigned to the sides and angles may be entirely different in an actual problem. Several applications of the law are sufficient to fix the law in mind, especially if the similarity of the law with the Pythagorean theorem is recognized. Bearing in mind that within a triangle the cosine of any angle greater than 90° is negative, the term $-2bc \cos A$, and so on, becomes positive if the included angle is over 90° . This makes the corre-

sponding value of a^2 larger than it would be if $A = 90^\circ$. Similarly, if the included angle is less than 90° , the term $-2bc \cos A$ remains negative, making the corresponding value of a^2 less than it would be if $A = 90^\circ$. For an angle of exactly 90° the term $-2bc \cos A$ becomes zero inasmuch as $\cos 90^\circ = 0$, and the cosine law for the particular angle is the same as the Pythagorean theorem. (Rule 186)

Three sides given:



$$81 = 36 + 144 - (2)(6)(12)(\cos A)$$

$$\cos A = \frac{99}{144} = \frac{11}{16} = .68750$$

$$A = 46^\circ 34.05'$$

$$36 = 144 + 81 - (2)(12)(9)(\cos B)$$

$$\cos B = \frac{189}{216} = \frac{7}{8} = .87500$$

$$B = 28^\circ 57.29'$$

$$144 = 36 + 81 - (2)(6)(9)(-\cos C)$$

$$\cos C = \frac{27}{108} = \frac{1}{4} = .25000$$

$$C = 104^\circ 28.64' \quad (\text{See explanation below})$$

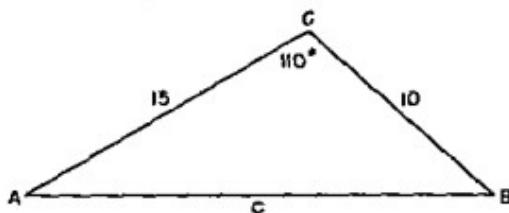
It should be observed that, by using the cosine law throughout the solution, the need for using computed values at any time is eliminated.

However one feature is involved which is troublesome if not understood. An inspection of the trigonometric tables shows that the angle which has a cosine of .25000 is not $104^\circ 28.64'$, but $75^\circ 31.36'$. Furthermore, the cosine of any angle in a triangle greater than 90° is always negative. Since the value obtained for $\cos C$ was positive, it should be an acute angle. An inspection of the given problem shows angle C to be greater than 90° , a fact easily proved by the Pythagorean theorem, or by subtracting the sum of A and B from 180° . If the unknown value of $\cos C$ had not been entered into the equation as a negative value, the result of the solution would have been $-.25000$ and have indicated that the angle was obtuse. This may be the better procedure, but since the value of the cosine of C was known to be negative, it was entered accordingly. The point of importance is not whether to enter $\cos C$ as $+$ or $-$, but to know whether the angle involved is less or greater than 90° . The angle corresponding to the computed function will be read from the tables as an acute angle. Finding the supplementary angle, if required, is then obtained by a simple subtraction of the acute angle from 180° .

Several variations of the above solution may be employed as shown in the following example. In this instance it is apparent that the sine proportion will not solve the

triangle as given, but is helpful after one of the unknown elements has been found by the cosine law.

Two sides and the included angle:



$$\cos 110^\circ = -\cos(180^\circ - 110^\circ) = -\cos 70^\circ = -.34202$$

$$c^2 = 225 + 100 - (2)(15)(10)(-.34202)$$

$$c^2 = 225 + 100 + 102.606 = 427.606$$

$$c = 20.678\dots = 20.68 \text{ (approx.)}$$

$$\sin 110^\circ = \sin(180^\circ - 110^\circ) = \sin 70^\circ = .93969$$

$$\frac{10}{\sin A} = \frac{20.68}{.93969}$$

$$\sin A = \frac{(10)(.93969)}{20.68} = .45440$$

$$A = 27^\circ 1.58'$$

$$\frac{15}{\sin B} = \frac{20.68}{.93969}$$

$$\sin B = \frac{(15)(.93969)}{20.68} = .68159$$

$$B = 42^\circ 58.10'$$

The above method of solution is more direct than if the cosine law had been employed to find the remaining elements. The use of the computed value for side (b), or 20.68 is unavoidable, regardless of the method employed.

The most common error in applying the cosine law is the failure to consider that the cosine of any angle in a triangle greater than 90° is negative. This causes the term $(-4xy)(\cos Z)$ to become $(-4xy)(-\cos Z)$ or $+4xy \cos Z$. After the solution is believed to be complete, an inspection to see that the greatest sides are opposite the largest angles often shows an error to exist.

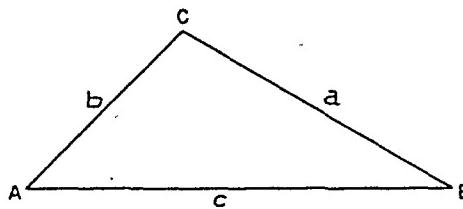
Law of Tangents

The sine proportions and the cosine law will solve any oblique triangle provided that a minimum of three elements are given, one of which must be a side. For slide rule, long hand, or calculating machine computations, these formulas are entirely satisfactory and should be used. However, if the triangles are to be solved by the use of logarithms, the cosine law is not a practical solution, since it is not expressed in terms of either ratios or products, and is therefore not adapted to use with logarithms. For this reason, the tangent law is stated and proved.

This formula is adapted to logarithmic computation and may be used to solve one of the types of oblique triangles solved by the cosine law, that is when two sides and the

included angle are given. The remaining case covered by the cosine law, three sides only, may be solved by logarithms by using a set of formulas, known as the half-angle formulas, which are tabulated on page 169.

The derivation of the tangent law is as follows:



$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } \frac{a}{b} = \frac{\sin A}{\sin B} \quad (\text{Rule 215})$$

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1$$

$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}$$

$$\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}$$

Similarly,

Dividing,

But,

$$\begin{aligned} \sin A - \sin B &= 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B) \\ &\quad (\text{Rule 249}) \end{aligned}$$

$$\begin{aligned} \sin A + \sin B &= 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) \\ &\quad (\text{Rule 248}) \end{aligned}$$

Dividing,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$\frac{a-b}{a+b} = \cot \frac{1}{2}(A+B) \tan \frac{1}{2}(A-B)$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

Again,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Therefore,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

Similarly,

$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}$$

When the first angle in the equation is greater than the second angle, the above formulas are used. To avoid negative quantities when the second angle is greater than the first, the formulas may be rewritten.

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}$$

$$\frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)}$$

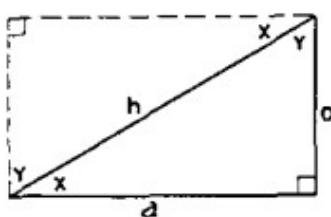
$$\frac{c-b}{c+b} = \frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}$$

The law of tangents may be stated:

The difference between two sides of a triangle is to their sum as the tangent of half the difference between the opposite angles is to the tangent of half the sum of these angles. (Rule 217)

TRIGONOMETRIC FORMULAS

There are a great number of formulas which are used at different times in solving trigonometric problems. A few of the most useful of these are derived in the following pages. Additional formulas are tabulated to provide a convenient reference.



$$x + y = 90^\circ$$

$$y = 90^\circ - x$$

$$\sin x = \frac{o}{h}$$

$$\cos y = \frac{b}{h}$$

$$\text{therefore, } \sin x = \cos y = \cos(90^\circ - x)$$

- (218). This relationship may be stated. The sine of any acute angle is equal to the cosine of its complementary angle
Also, $\cos x = \sin y = \sin(90^\circ - x)$

- (219). This relationship may be stated. The cosine of any acute angle is equal to the sine of its complementary angle.
Similarly,

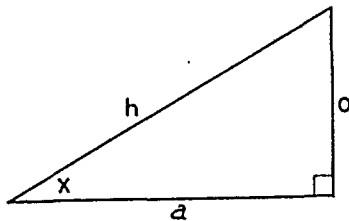
$$(220). \quad \tan x = \cot y = \cot(90^\circ - x)$$

$$(221). \quad \operatorname{cosec} x = \sec y = \sec(90^\circ - x)$$

$$(222). \quad \sec x = \operatorname{cosec} y = \operatorname{cosec}(90^\circ - x)$$

$$(223). \quad \cot x = \tan y = \cot(90^\circ - x)$$

Relations Between the Functions of One Angle



$$\sin x = \frac{o}{h}$$

$$\cos x = \frac{a}{h}$$

$$\frac{\sin x}{\cos x} = \frac{o/h}{a/h} = \frac{o}{a}$$

but, $\tan x = \frac{o}{a}$

(224). Therefore, $\frac{\sin x}{\cos x} = \tan x$

(225). $\frac{\cos x}{\sin x} = \cot x$

$$o^2 + a^2 = b^2$$

$$\frac{o^2}{b^2} + \frac{a^2}{b^2} = \frac{b^2}{b^2} = 1$$

but, $\frac{o}{b} = \sin x$ $\frac{o^2}{b^2} = \sin^2 x$

$\frac{a}{b} = \cos x$ $\frac{a^2}{b^2} = \cos^2 x$

(226). Therefore, $\sin^2 x + \cos^2 x = 1$

To indicate a power of a trigonometric function, it is customary to apply the exponent directly to the function, rather than to write the exponent after the angle. Thus,

$\sin^2 x$ means $(\sin x)^2$.

The expression $\sin x^2$ would mean the sine of an angle whose number of units is the square of the number in angle (x).

Relations Between the Sum and Difference of Two Angles

The sum of any two angles such as (x) and (y) can be denoted as (s), and the difference between the same two angles can be denoted as (d). Assume that (x) is the larger angle.

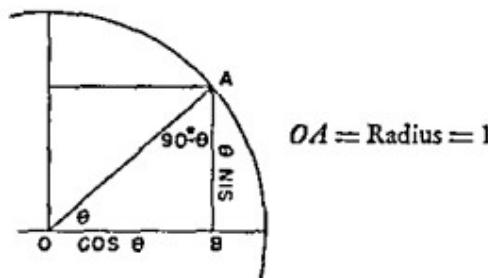
$$\begin{array}{r} x + y = s \\ x - y = d \\ \hline 2x = s + d \\ x = \frac{1}{2}(s + d) \\ x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \end{array}$$

(227). or,

$$\begin{array}{r} x + y = s \\ x - y = d \\ \hline 2y = s - d \\ y = \frac{1}{2}(s - d) \\ y = \frac{1}{2}(x + y) - \frac{1}{2}(x - y) \end{array}$$

(228). or,

By constructing a circle with a radius (OA) of unity, and employing the definitions of sin and cos as the lengths of certain lines within that circle, the three preceding relationships may be verified.



$$(224). \quad \tan \theta = AB/OB = \sin \theta / \cos \theta$$

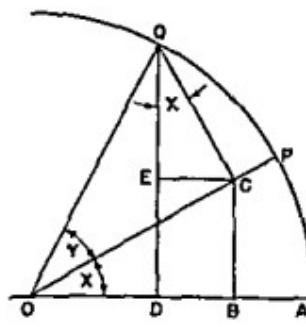
$$(218). \quad \sin \theta = \cos(90^\circ - \theta)$$

$$(219). \quad \cos \theta = \sin(90^\circ - \theta)$$

$$\sin^2 \theta + \cos^2 \theta = (OB)^2 = 1^2 = 1$$

$$(226). \quad \sin^2 \theta + \cos^2 \theta = 1$$

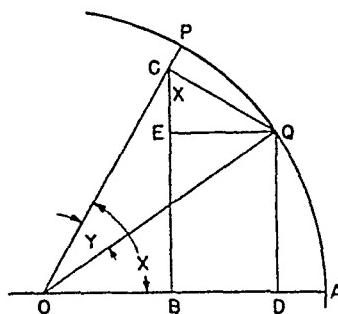
In the figure below, let (x) and (y) be any two acute angles each of which, as well as their sum, is less than 90° . QC is perpendicular to OP . BC and DQ are perpendicular to OA , and EC is parallel to OA . The radius of the circle (OA , OP , or OQ) is equal to unity.



$$\begin{aligned}
 x + y &= A O Q \\
 \text{angle } EQC &= x; OC = \cos y; CQ = \sin y \\
 \sin(x + y) &= DQ = BC + EQ \\
 &= OC \sin BOC + CQ \cos EQC \\
 &= \cos y \sin x + \sin y \cos x \\
 \sin(x + y) &= \sin x \cos y + \cos x \sin y \\
 \cos(x + y) &= OD = OB - EC \\
 &= OC \cos BOC = CQ \sin EQC \\
 &= \cos y \cos x - \sin y \sin x \\
 \cos(x + y) &= \cos x \cos y - \sin x \sin y
 \end{aligned}
 \tag{229}.$$

$$(230).$$

In the figure below, let (x) and (y) be any two acute angles each of which is less than 90° and (x) being greater than (y) . QC is perpendicular to OP . BC and DQ are perpendicular to OA , and EQ is parallel to OA . The radius of the circle (OA , OQ , or OP) is equal to unity.



$$x - y = A O Q$$

$$\begin{aligned}
 \text{angle } ECQ &= x; OC = \cos y; CQ = \sin y \\
 \sin(x - y) &= DQ = BC - EC \\
 &= OC \sin BOC - CQ \cos ECQ \\
 &= \cos y \sin x - \sin y \cos x \\
 \sin(x - y) &= \sin x \cos y - \cos x \sin y \\
 \cos(x - y) &= OD = OB + EQ \\
 &= OC \cos BOC + CQ \sin ECQ \\
 &= \cos y \cos x + \sin y \sin x \\
 \cos(x - y) &= \cos x \cos y + \sin x \sin y
 \end{aligned}
 \tag{231}.$$

$$(232).$$

$$\begin{aligned}
 \cos(x - y) &= \cos x \cos y + \sin x \sin y
 \end{aligned}$$

Although equations (229)–(232) were derived for acute angles only, and other special requirements as noted, it may be shown that the equations are true for all values of these angles by proving the special cases separately. Throughout the foregoing derivations, it must be remembered that the lines representing the functions are the actual trigonometric functions since the radius of the circle is unity.

The derivation of the tangent of the sum and the difference of two angles is accomplished algebraically without resort to the geometric proof:

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Divide both numerator and denominator by $\cos x \cos y$:

$$\begin{aligned} \tan(x+y) &= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}} \\ (233). \quad \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \end{aligned}$$

In the same way it may be shown:

$$(234). \quad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Functions of an Angle in Terms of those of Half the Angle

$$\begin{aligned} (235). \quad \sin 2x &= 2 \sin x \cos x \\ (236). \quad \cos 2x &= \cos^2 x - \sin^2 x \\ (237). \quad \cos 2x &= 1 - 2 \sin^2 x \\ (238). \quad \cos 2x &= 2 \cos^2 x - 1 \\ (239). \quad \tan 2x &= \frac{2 \tan x}{1 - \tan^2 x} \end{aligned}$$

Functions of an Angle in Terms of those of Double the Angle

$$\begin{aligned} (240). \quad 2 \sin^2 x &= 1 - \cos 2x \\ (241). \quad 2 \cos^2 x &= 1 + \cos 2x \\ (242). \quad \tan^2 x &= \frac{1 - \cos 2x}{1 + \cos 2x} \\ (243). \quad \tan x &= \frac{\sin 2x}{1 + \cos 2x} \\ (244). \quad \tan x &= \frac{1 - \cos 2x}{\sin 2x} \end{aligned}$$

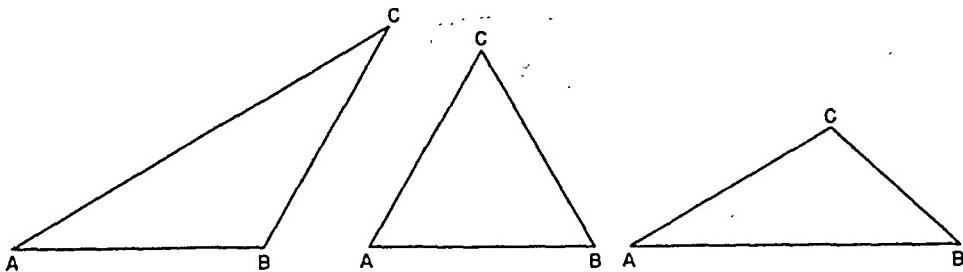
The Product of Functions of Angles in Terms of Sums of Functions

$$\begin{aligned} (245). \quad \sin x \cos y &= \frac{1}{2} \sin(x+y) + \frac{1}{2} \sin(x-y) \\ (246). \quad \cos x \sin y &= \frac{1}{2} \sin(x+y) - \frac{1}{2} \sin(x-y) \\ (247). \quad \cos x \cos y &= \frac{1}{2} \cos(x+y) + \frac{1}{2} \cos(x-y) \\ (248). \quad \sin x \sin y &= -\frac{1}{2} \cos(x+y) + \frac{1}{2} \cos(x-y) \end{aligned}$$

The Algebraic Sum of Functions of Angles in Terms of the Product of Functions

$$\begin{aligned} (249). \quad \sin x + \sin y &= 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) \\ (250). \quad \sin x - \sin y &= 2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) \\ (251). \quad \cos x + \cos y &= 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) \\ (252). \quad \cos x - \cos y &= -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) \end{aligned}$$

Oblique Triangle Formulas



$$(215). \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{Sine Proportion}$$

$$(216). \quad \begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned} \quad \text{Cosine Law}$$

$$(217). \quad \begin{aligned} \frac{a-b}{a+b} &= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} \\ \frac{a-c}{a+c} &= \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)} \\ \frac{b-c}{b+c} &= \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} \end{aligned} \quad \text{Tangent Law}$$

(253).

$$\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin \frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$\sin \frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Half-angle formulas, where
(s) is half the sum of the
three sides and:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

(254).

$$\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\cos \frac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}$$

$$\cos \frac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}$$

(255).

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\text{or, } \tan \frac{1}{2}A = \frac{r}{s-a}$$

$$\tan \frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\text{or, } \tan \frac{1}{2}B = \frac{r}{s-b}$$

$$\tan \frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

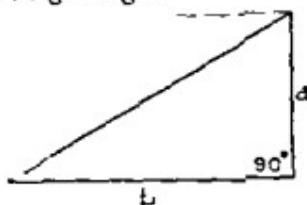
$$\text{or, } \tan \frac{1}{2}C = \frac{r}{s-c}$$

SOLUTION OF EQUATIONS

AREA OF TRIANGLES

Right Triangle

(256). The area of a right triangle is equal to one-half the product of the two sides forming the right angle.

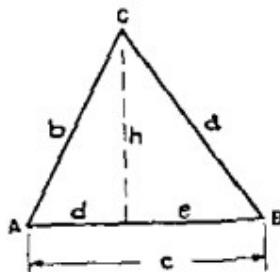


$$\text{Area} = \frac{1}{2} (a) (b) = \frac{ab}{2}$$

Oblique Triangle

(257). The area of an oblique triangle is equal to one-half the product of any two sides and the sine of an included angle.

$$\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B$$



$$\text{Area} = \frac{1}{2} db + \frac{1}{2} eb = \frac{1}{2} b(d + e)$$

$$= \frac{1}{2} bc$$

$$b = b \sin A \quad \text{also, } b = a \sin B$$

$$\text{Area} = \frac{1}{2} bc \sin A$$

$$\text{Area} = \frac{1}{2} ac \sin B$$

Similarly,

$$\text{Area} = \frac{1}{2} ab \sin C$$

Another formula makes possible the finding of the area of any triangle when the lengths of the three sides are given.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

(258).

$$\text{Where, } s = \frac{1}{2}(a+b+c) \quad (a+b+c) = \text{perimeter}$$

Section IV

LOGARITHMS

INTRODUCTION

The invention of logarithms was one of the great inventions of all times for the saving of time and labor. Many calculations which are difficult or impossible by other methods are readily made by means of logarithms. Using logarithms, the process of multiplication is replaced by one of addition; that of division, by one of subtraction; that of raising to a power, by a simple multiplication; and that of extracting a root becomes division.

Although all of the operations enumerated above can be performed using logarithms, the processes of multiplication and division are not ordinarily solved by this method as long-hand methods are simpler, more accurate, and already understood by most people. Furthermore, with the development of the modern calculating machine, a means for obtaining products and quotients has been provided which is superior to either the use of logarithms or ordinary long-hand computation for those specified operations. However, for obtaining roots and powers of numbers, the use of logarithms is not only desirable, but a necessity in many cases as most fractional roots and powers can be determined only by this means.

Before making practical use of the power and convenience of logarithms it is advisable to learn something of the principle upon which the system is based.

DEFINITIONS AND PRINCIPLES

There are four fundamental rules for operations with exponents, namely:

$$\text{Multiplication, } N^a \times N^b = N^{a+b}$$

or, the product of two or more like quantities (which have either like or unlike exponents) is equal to this quantity with the sum of the exponents of the factors as an exponent.

$$\text{Division, } N^a \div N^b = N^{a-b}$$

or, the quotient of two like quantities (which have either like or unlike exponents) is equal to this quantity with the difference between the exponents of the dividend and the divisor as an exponent.

$$\text{Power, } (N^a)^b = N^{ab}$$

or, any quantity raised to any power is equal to the quantity with its original exponent multiplied by the power in question.

$$\text{Root, } \sqrt[b]{N^a} = (N^a)^{\frac{1}{b}} = N^{\frac{a}{b}}$$

or, any quantity from which a given root is to be extracted is equal to the quantity with its exponent divided by the root in question.

If any number, such as 2, is taken as the number N referred to above, and various exponents from 0 to 10 are assumed, the following table can be constructed:

SOLUTION OF EQUATIONS

N	Exponent	Value
2	0	1
2	1	2
2	2	4
2	3	8
2	4	16
2	5	32
2	6	64
2	7	128
2	8	256
2	9	512
2	10	1024

This table can be used to perform the four fundamental operations with exponents by representing the numbers involved as some power of the number 2. For example:

Multiplication,

$$8 \times 64 = 2^3 \times 2^6 = 2^9 = 512$$

Division,

$$\frac{1024}{16} = 2^{10} - 2^4 = 2^6 = 64$$

Power,

$$16^2 = (2^4)^2 = 2^8 = 256$$

Root,

$$\sqrt[3]{256} = \sqrt[3]{2^8} = 2^{\frac{8}{3}} = 2^4 = 16$$

In the small table of the powers of 2 given above there are many gaps, because only those powers which have *whole* exponents are given. For all the numbers between 16 and 32, for example, the exponents will be decimals, and will be greater than 4 but less than 5, etc. In practice, the base used is not 2, but 10, and all the intermediate exponents have been computed to many decimals, these forming a table of logarithms.

If 10 is taken as a base and (a) as an exponent, then, for any number,

$$N = 10^a$$

The exponent (a) is the logarithm of (N) when the base is 10, or, the *logarithm* of number is the exponent of a power to which a certain number, called the *base*, must be raised to produce the given number. When the base number is not specified, it is generally understood to be 10, and the logarithms to this base are termed *common* logarithms. Thus the logarithm (common) of a number is the power to which 10 must be raised to equal the number. For common logarithms.

$$\begin{aligned} 1000 &= 10^3 \\ \log 1000 &= 3.000000 \\ 100 &= 10^2 \\ \log 100 &= 2.000000 \\ 10 &= 10^1 \\ \log 10 &= 1.000000 \end{aligned}$$

LOGARITHMS

$$1 = \frac{10}{10} = \frac{10^1}{10^1} = 10^0$$

$$\log 1 = 0.000000$$

$$0.1 = \frac{1}{10} = \frac{1}{10^1} = 10^{-1}$$

$$\log 0.1 = -1.000000$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$$

$$\log 0.01 = -2.000000$$

$$0.005 = \frac{5}{1000} = \frac{1}{200} = \frac{1}{10x} \quad (\text{where } x > 2 \text{ but} < 3)$$

$$= \frac{1}{10^{2+c}} = 10^{-2-c} = 10^{-3+m}$$

(See below)

$$\log 0.005 = -3 + .698970 = \overline{3}.698970 \quad (\text{see below})$$

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$$

$$\log 0.001 = -3.000000$$

From the foregoing, it is evident that only a very few numbers have logarithms which are integers, that is, whole numbers without any accompanying fraction. For instance, the logarithms of any number between 100 and 1000 will have a value greater than +2, yet less than +3, since +2 is the logarithm of 100 and +3 is the logarithm of 1000. Such a logarithm is written, $(2 + m)$ where (m) represents a positive decimal fraction called the *mantissa*. The integer portion of the logarithm, which in this case is +2, is termed the *characteristic*. The complete logarithm of the number is, therefore, $(2 + m)$, or the sum of the characteristic and the mantissa.

The explanations and solutions of examples throughout SECTION IV are based on data obtained from a six-place table of logarithms of numbers and a six-place table of logarithms of the trigonometric functions. However, the same procedure is employed if tables of either less or greater accuracy are employed.

N		6		
345		.538574		

The complete logarithm of a number is not obtained directly from a table of so-called logarithms because such tables are generally tabulations of mantissas only, the values of the characteristics being omitted altogether. The tables are then actual logarithms only for the numbers ranging in value from 1.0 to 9.99 inclusive, since these numbers have zero as a characteristic. (See RULES FOR CHARACTERISTICS). In finding

the mantissa part of the logarithm, no attention is given to the position of the decimal point in the given number because the mantissa is always identical for the same sequence of figures. For example, the mantissa part of the logarithm of 3.456 is identical to the mantissa part of the logarithm of 345.6, the value being .538574 in both cases. Rules for finding the value of the characteristic of any given number are given elsewhere. If used to determine the characteristic of 345.6, the value will be found to be 2, which will verify the statement given above that the characteristic of the logarithm of any number between 100 and 1000 is 2. The complete logarithm of 345.6 is therefore $2 + .538574$ or 2.538574.

The use of a so-called logarithm table for finding the mantissa part of the logarithm of a number greater than unity has already been described. The tables are similarly used in finding the mantissa part of the logarithm of a number less than unity, that is a decimal fraction. However, it should be thoroughly understood that the tabulated mantissas are all positive in sign and when dealing with decimal fractions the characteristic will be negative and the mantissa positive. Great care must be exercised in performing subsequent operations with such logarithms. That the characteristic will be negative and the mantissa positive for any given decimal fraction may be shown by an example. From Page 173 it is evident that the logarithm of 0.005 is a number between -2 and -3. This is true because the number 0.005 is in between .01 and .001 whose logarithms are -2 and -3 respectively. Since 0.005 is smaller numerically than .01, being only one-half as large, its logarithm will be more negative than the logarithm of .01 since the value of all logarithms increase negatively as the values of the numbers become smaller. The actual value of the logarithm of .005 is -2.301030 which can also be written as -2.000000 - 0.301030. The value of the mantissa of this logarithm (-0.301030) is *not* given in the tables because the mantissas contained therein are always positive. To express the complete logarithm of .005 as a number in which the mantissa is positive it is possible to add 1 to the mantissa and subtract the same 1 from the characteristic. This is obviously a valid algebraic operation since

$$\begin{aligned} -2.301030 &= -2.000000 - 0.301030 - 1 + 1 \\ &= -2.000000 - 1 - 0.301030 + 1 \\ &= -3.000000 + .698970 \\ &= \overline{3} .698970 \end{aligned}$$

The logarithm as written $\overline{3} .698970$ with the negative sign placed over the characteristic denotes that it *alone* is negative. Thus, $\overline{3} .698970$ means $-3.000000 + .698970$ or -2.301030 . The fallacy of writing the $\log 0.005 = -3.698970$ is apparent for the minus sign would indicate that both the characteristic and the mantissa are negative. Of course, the complete logarithm of a decimal fraction can be written as an entirely negative number as shown above, but then the mantissa part of the logarithm does not correspond to the mantissa as given in a table of logarithms where all mantissas are *always* positive. The facts as described above are the basis for the rule concerning the value of the characteristic of the logarithms of decimal fractions.

The complete logarithm of any number, decimal fractions or otherwise, is obtained by determining the characteristic and adding to it or annexing the value of the mantissa. The reverse order of determining the mantissa first and prefixing the characteristic is also valid since it produces the same result. The absolute value and algebraic sign of

the characteristic depends upon the position of the decimal point in the given numbers, and is easily found by either one of the two methods to follow. Mantissas are obtained directly from the table unless the given number consists of more than four digits, in which case interpolation is required, (see page 177).

RULES FOR CHARACTERISTICS

The characteristic for any number is found by either of the following methods.

Method 1.—When the given number is *greater* than one, the characteristic is positive and is one *less* than the number of places (digits) appearing to the left of the decimal point in the given number. When the given number is *less* than one, the characteristic is *negative* and is one *more* than the number of zeros between the decimal point and the first significant figure in the given number.

Method 2.—A more simple method is to remember that the characteristic is zero for any given number which has but a single place (digit) to the left of the decimal point. Moving the decimal point to the right increases the characteristic by one for each place moved. Moving the decimal point to the left decreases the characteristic (increases the negativity) by one for each place moved.

AUGMENTED LOGARITHMS

To avoid the use of negative characteristics such as $\bar{3}.301030$ it is possible to add any quantity to the logarithm provided that it is indicated that the *same* quantity is also to be subtracted.

$$\begin{aligned}\text{Log .002} &= \bar{3}.301030 = -3.000000 + .301030 \\ &= -3.000000 + 10 + .301030 - 10 \\ &= +7.000000 + .301030 - 10 \\ &= 7.301030 - 10\end{aligned}$$

The result can be obtained directly by adding a $+10$ to the characteristic, and from the resulting logarithm subtracting 10.

$$\bar{3}.301030 = 7.301030 - 10$$

The validity of this operation is further established by reducing both the given logarithm $\bar{3}.301030$ and its equivalent form $7.301030 - 10$ to their respective values as represented by single sequences of numbers.

$$\begin{aligned}\bar{3}.301030 &= -3.000000 + .301030 = -2.698970 \\ 7.301030 - 10 &= -10.000000 + 7.301030 = -2.698970\end{aligned}$$

Logarithms operated on as described above are termed augmented logarithms. Other values, usually in multiples of 10, are sometimes used in place of 10. The reason for using 10 or some multiple of 10 is to eliminate errors in subtraction, because it is easier to subtract 10, 20, or 30 from any number than it is to subtract any other value. However, any number may be so employed provided that the negative characteristics of the given logarithm becomes positive and of such absolute value that when the number is subtracted from the characteristic, the original negative characteristic is regained.

SOLUTION OF EQUATIONS

$$\begin{aligned} \bar{3.698970} &= 7.698970 - 10 \\ &= 17.698970 - 20 \\ &= 0.698970 - 3 \\ &= 3.698970 - 6 \end{aligned}$$

Another operation which cannot be strictly classified as augmenting or enlarging a logarithm is described in this section for convenience. This operation consists of changing a logarithm having a negative characteristic and a positive mantissa into an entirely negative number so that the resulting number may be used as a multiplier, or divisor. For example, the logarithm of 0.002 is $\bar{3}301030$, which means $-3 + 0.301030$. When written as $\bar{3}.301030$ the number cannot be used either as a multiplier or a divisor, but if the absolute value of the logarithm is determined, then it may be so used. The true value of the expression $\bar{3}301030$ is $-3.000000 + 0.301030$ or,

$$\begin{array}{r} -3\ 000000 \\ 0.301030 \\ \hline -2\ 698970 \end{array}$$

These operations are not confined to logarithms having negative characteristics as implied above. The same procedure may be employed to advantage in solving problems in which logarithms having positive characteristics are involved. The example below is solved by employing the previously described principles, thus avoiding, as a result of the indicated operation, a logarithm which is entirely negative.

$$\begin{aligned} \log a &= \log 24 - \log 48 \\ &= 1.380211 - 1.681241 \\ &= (11.380211 - 10) - 1.681241 \\ &\quad \begin{array}{r} 11.380211 - 10 \\ 1.681241 \\ \hline 9.698970 - 10 \end{array} \\ \log a &= \bar{1}.698970 \\ a &= 0.50 \end{aligned}$$

The reverse procedure of changing an entirely negative logarithm (both characteristic and mantissa negative) into a logarithm with a positive mantissa is required before its anti-logarithm can be found. This requirement is evident if it is remembered that all mantissas tabulated in a table of logarithmic functions are positive in sign.

$$\log x = -0.301030$$

In this case the characteristic is zero and the value of the mantissa is -0.301030 . This same value may be represented by the sum of two terms, one of which is negative and the other positive.

$$\begin{aligned} -0.301030 &= -1.000000 + .698970 \\ &= \bar{1}.698970 \end{aligned}$$

The result can be obtained directly by adding $+1$ to the mantissa making it a positive decimal and subtracting -1 from the characteristic. The sum of $+1$ and any negative mantissa regardless of the sequence of numbers can be written down directly by

subtracting each number of the given mantissa from 9, except the right-hand digit which is subtracted from 10.

$$\log x = -3.456789 = \bar{4}.543211$$

USE OF TABLES OF LOGARITHMS—INTERPOLATION

Interpolation of Mantissas

The mantissa of a number of more than four digits can be found to a high degree of accuracy by assuming that the change in the value of the mantissa is directly proportional to the change in the value of the number. This relationship actually does not exist inasmuch as it would require, for example, that the mantissa of 50 should be twice the mantissa of 25. The actual values are .698970 and .397940 respectively, which are obviously not in the ratio of 2 to 1. One exception to this reasoning might at first seem to be justified in the case of the mantissas of 4 and 2 which are .602060 and .301030 respectively. However, the value of the tabular difference for 200 is 217, and for 400 is 109, which indicates that the change in the value of the mantissa is *not* proportional to the change in value of the number.

The error resulting from the above assumption is of small consequence in any ordinary computation because it is only between the tabulated values of the mantissa that the assumption is assumed to apply, and not to the table as a whole.

The following steps are employed in finding the mantissa of a number consisting of five or more digits. It is assumed that an ordinary logarithm table is being used which allows the mantissa of a number of four or fewer digits to be read directly.

Step 1. Find the mantissa corresponding to the first four digits of the number.

Step 2. Multiply (A) the tabular difference between the mantissa obtained in step 1 and the mantissa adjacent and next higher in the tables by (B) the fifth and following digits of the given number treated as a decimal fraction.

Step 3. Add the product obtained in step 2 to the mantissa obtained in step 1. The sum will be the desired mantissa. The number of digits retained in the interpolated value should not be greater than the number of digits occurring in the tables being used.

Mantissa 34567

N.		6	7	Diff.
345		.538574	.538699	126

SOLUTION OF EQUATIONS

$$\text{Mantissa of } 3456 = 538574 \quad (\text{Step 1.})$$

$$\text{Mantissa of } 3457 = 538699$$

$$\text{Mantissa of } 3456 = \underline{\underline{538574}}$$

$$\text{Tabular difference} = .000125$$

$$(000125) (7) = 0000875$$

(Step 2.)

$$= 000087$$

$$\text{or, } = 000088$$

(Either this value or the one below it may be used.)

$$\text{Mantissa of } 3456 = 538574$$

$$\underline{.000088}$$

(Step 3.)

$$\text{Mantissa of } 3456 = .538662$$

Considerable effort can be spared if the tabular differences are not computed in each case as indicated in step 2. This subtraction is unnecessary if the difference column to the right of each page of the tables is employed. This column indicates the *average difference*, with no regard to decimal point, between the mantissas of any two columns on the same line. This tabulated average difference corresponds to the exact difference in most cases where large numbers are involved and the value of the mantissa is changing slowly. For small values the last digit of the tabulated difference should be checked against the actual difference of the last digits of the two mantissas if the maximum accuracy is desired. In the specific example just solved the average tabular difference is shown as 126, while the actual tabular difference is 125.

Interpolation in Finding Numbers

After a logarithm or number of logarithms have been operated on as described in the paragraphs entitled FUNDAMENTAL OPERATIONS USING LOGARITHMS, the next and final operation is to find the numerical value of the answer which is represented by its logarithm. Such a number is termed the anti-logarithm of the logarithm, or simply anti-log. The anti-log is obviously the answer in ordinary numbers, and is found by process just the reverse of finding the logarithm.

In finding the logarithm of a number the process of interpolation is required in only a fraction of the total number of times because numbers of four or fewer digits are most frequently involved. Unfortunately, in finding anti-logarithms the need for interpolation is the rule rather than the exception because the given mantissa is rarely found to correspond to a mantissa included in the tables. Where interpolation is necessary the procedure is exactly a reverse process of that previously described for finding mantissae. This procedure is most easily explained by an example.

$$\log x = 337110$$

$$x =$$

N		3	4		Diff.
217		.337060	.337260		200

Step 1. From the tables it is evident that the given mantissa has a value which lies in between .337060 and .337260. These values of mantissa correspond to the sequence of numbers 2173 and 2174 respectively. The sequence of numbers corresponding to a mantissa of .337110 is therefore 2173 +.

Step 2. The given mantissa exceeds the mantissa of 2173 by .000050.

$$\begin{array}{rcl} \text{Mantissa of given number} & = & .337110 \\ \text{Mantissa corresponding to 2173} & = & \underline{.337060} \\ \text{Difference} & = & .000050 \end{array}$$

And, the mantissa of 2174 exceeds the mantissa of 2173 by 200.

$$\begin{array}{rcl} \text{Mantissa corresponding to 2174} & = & .337260 \\ \text{Mantissa corresponding to 2173} & = & \underline{.337060} \\ \text{Tabular difference} & = & .000200 \end{array}$$

Assuming that the change in the value of the mantissa is directly proportional to the change in the value of the number, the fifth and succeeding numbers of the sequence 2173 can be determined by the ratio whose numerator is the amount by which the value of the given mantissa exceeds the mantissa of 2173, and whose denominator is the total tabular difference of the mantissas of 2173 and 2174 respectively.

$$\frac{50}{200} = .25$$

Step 3. The sequence of figures is therefore:

217325

Since the given logarithm had a characteristic of zero,

$$x = 2.17325$$

The interpolation fraction is rarely of such magnitude that division can be performed exactly. In most cases the quotient will be an irrational number allowing no exact solution. For any ordinary problem the results obtained by performing this division with a slide rule are satisfactory. (See page 205).

FUNDAMENTAL OPERATIONS USING LOGARITHMS

The common logarithm of a number is the power to which the number 10 must be raised to equal the given number. Therefore, since logarithms are exponents, the operations involving logarithms are governed by the same rules that apply to exponents. As stated on the first page of this section, logarithms may be used in all computations except addition and subtraction. The operations of multiplication and division are more easily performed by other methods, but for the operations involving roots and powers of most numbers the use of logarithms is desirable, and, in a great number of cases, an absolute necessity.

The detailed description which has been given in the previous paragraphs may make it appear that the amount of labor involved in the use of logarithms is considerable. Actual application, however, will demonstrate the brevity and simplicity of the system, especially if exact interpolation is not required.

SOLUTION OF EQUATIONS

Multiplication

$$(10^a) (10^b) = 10^{a+b} \quad (\text{Rule 29})$$

- (259) The logarithm of the product of two or more quantities is equal to the sum of the logarithms of the factors being multiplied together. The anti-logarithm of this sum is the product of the several factors.

$$\begin{aligned}x &= (a)(b) \\ \log x &= \log(a) + \log(b) \\ x &= \text{anti-log}[\log(a) + \log(b)] \\ x &= (391)(375) \\ \log x &= \log 391 + \log 375 \\ &= 2.592177 + 2.574031 \\ \log x &= 5.166208 \\ x &= 146625.00\end{aligned}$$

Division

$$10^a \div 10^b = 10^{a-b} \quad (\text{Rule 31})$$

- (260) The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor. The anti-logarithm of this difference is the quotient.

$$\begin{aligned}x &= a/b \\ \log x &= \log(a) - \log(b) \\ x &= \text{anti-log}[\log(a) - \log(b)] \\ x &= \frac{48}{24} \\ \log x &= \log 48 - \log 24 \\ &= 1.681241 - 1.380211 \\ \log x &= 0.301030 \\ x &= 2.00\end{aligned}$$

Powers

$$(10^a)^n = 10^{an} \quad (\text{Rule 34})$$

- (261). The logarithm of any power of any number is equal to the product of the power (exponent) and the logarithm of the number. The anti-logarithm of this product is the value of the number raised to the power in question.

$$\begin{aligned}x &= a^n \\ \log x &= n \log a \\ x &= \text{antilog}(n \log a) \\ x &= 25^3 \\ \log x &= 3 \log 25 = 3(1.397940) \\ \log x &= 4.193820 \\ x &= 15625.00\end{aligned}$$

The multiplication of (3) (1.397940) in the solution above could have been solved by logarithms, and in problems where both factors are of many digits such an operation would result in a great saving of time. Using the above example;

$$\begin{aligned}\log x &= (3)(1.397940) \\ \log(\log x) &= \log 3 + \log 1.397940 \\ &= 0.477121 + 0.145488 = 0.622609 \\ \log x &= 4.193816 \\ x &= 15624.86\end{aligned}$$

Roots

$$\sqrt[n]{10} = (10^{\circ})^{1/n} = 10^{\circ/n} \quad (\text{Rule 35})$$

- (262). The logarithm of any root of any number is equal to the logarithm of the number divided by the root in question. The antilogarithm of this quotient is the value of the required root.

$$x = \sqrt[n]{a}$$

$$\log x = \frac{1}{n} \log a \text{ or } \frac{\log a}{n}$$

$$x = \sqrt{625}$$

$$\log x = \frac{1}{2} \log 625 = \frac{\log 625}{2} = \frac{2.795880}{2}$$

$$\begin{aligned}\log x &= 1.397940 \\ x &= 25.000\dots\end{aligned}$$

The four basic operations which can be performed by logarithms are all demonstrated in the previous paragraphs. These procedures must be thoroughly understood before attempting the same operations with less simple numbers, and in performing several operations all within one equation.

COLOGARITHMS

It has been shown in Section 1, (Rule 176) that the quotient obtained when (a) is divided by (b) is the same as the product of (a) and the reciprocal of (b) , or:

$$\left(\frac{a}{b}\right) = \frac{1}{b}(a)$$

When this principle is applied to logarithmic operation it is possible to perform an indicated division by adding the logarithm of the dividend (numerator) to the logarithm of the reciprocal of the divisor (denominator).

$$\log\left(\frac{a}{b}\right) = \log\left(\frac{1}{b}\right) + \log a$$

The logarithm of the reciprocal of any number such as (b) is $\log(1/b)$ and is numerically the same as $-\log b$.

$$\log\left(\frac{1}{b}\right) = \log 1 - \log b$$

but

$$\log 1 = 0$$

$$\log\left(\frac{1}{b}\right) = 0 - \log b = -\log b$$

The logarithm of the reciprocal of a number has been given the specific title of cologarithm. This title is appropriate since in converting a cologarithm (which is the negative logarithm of the number) to a logarithm having a positive mantissa, the given mantissa is added to 1 which gives a positive decimal whose absolute value is the complement of the original negative characteristic. Since +1 has been added to the mantissa of the logarithm, it is necessary to subtract 1 from the characteristic. These operations have already been described in detail on page 175.

$$x = \frac{1}{12}$$

$$\log x = \log \left(\frac{1}{12} \right) = \text{colog } 12 = -\log 12$$

DIVISION OR MULTIPLICATION OF LOGARITHMS

There are some special cases in which one logarithm is to be either multiplied or divided by another logarithm. Such operations are not very frequent and one rarely becomes well enough acquainted with this type of problem to be at all certain of the validity of the results obtained. In many instances the answer will not be correct because of an erroneous procedure, specifically, the addition of logarithms instead of multiplication, or the subtraction of logarithms instead of division. This fallacy may be partially eliminated by writing the solution of the problem in algebraic form which is a recommended procedure in any solution, and then following out the operations indicated by the equation

$$2^x = 25 \\ x \log 2 = \log 25$$

$$(x) (0.301030) = 1.397940$$

$$x = \frac{1.397940}{.301030} = 4.6438$$

Note that x does not equal $1.397940 - .301030$

$$\text{The equation says } \frac{1.397940}{.301030} \quad (\text{Rule 40})$$

The result is apparently correct since $2^4 = 16$ and $2^5 = 32$. In dividing the two logarithms to obtain the quotient, 4.6438, it would have been possible to employ logarithms inasmuch as this would be equivalent to the example shown on page 180. However, actual division seems to be the more practical solution in this case.

$$\frac{.1875}{.750} = \left(\frac{24}{48} \right)^x$$

This problem can be solved by inspection as the value of (x) is obviously 2. However, such convenient problems do not exist in actual practice. The solution of the problem assumes that the given ratios are of such value that they cannot be simplified other than by the operations shown.

$$\frac{750}{187.5} = \left(\frac{48}{24}\right)^x$$

(Rules 9, 41)

$$\log 750 - \log 187.5 = x(\log 48 - \log 24)$$

$$x = \frac{\log 750 - \log 187.5}{\log 48 - \log 24} = \frac{2.875061 - 2.273001}{1.681241 - 1.380211}$$

$$x = \frac{0.602060}{0.301030} = 2$$

Note that $\log 48 - \log 24$ does not equal $\log(48 - 24)$. The difference in the logs of two numbers is not equal to the log of the difference of the two numbers. This is often erroneously assumed to be true.

SOLUTION OF EQUATIONS USING LOGARITHMS

Complex equations involving several operations are solved by proper applications of Rules 259, 260, 261, and 262. The writing of the solutions in algebraic forms is recommended in all cases.

$$\begin{aligned} 133.5 &= P(6)^{1.3} \\ \log 133.5 &= \log P + 1.3 \log 6 \\ 2.125481 &= \log P + 1.3(0.778151) \\ 2.125481 - 1.011596 &= \log P \\ \log P &= 1.113885 \\ P &= 12.9983 \end{aligned}$$

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334

NOTE: The fraction appearing within the rectangular square is the interpolation fraction used in obtaining the final result. This information is not always shown as a part of the solution since the actual interpolation is often performed by the use of a slide rule or a table of proportional parts. The interpolation fraction is included in this and the following problems so that these solutions may serve as additional examples to those already given in the paragraphs entitled Interpolation in Finding Numbers.

$$D = 300 \sqrt[4]{\frac{550}{(2200)^2 (182)}}$$

$$\begin{aligned} \log D &= \log 300 + 1/4(\log 500 - 2 \log 2200 - \log 182) \\ &= 2.477121 + 1/4 [2.740363 - 2(3.343423) - 2.260071] \\ &= 2.477121 + 1/4 (2.740363 - 6.684846 - 2.260071) \\ &= 2.477121 - \frac{6.204554}{4} = 2.477121 - 1.551139 \end{aligned}$$

$$= .925982$$

$$D = 8.4330$$

$$175 = (K)(2375')$$

$$150 = (K)(2100')$$

SOLUTION OF EQUATIONS

$$\log 175 = \log K + x \log 2375$$

$$\log 150 = \log K + x \log 2100$$

$$\underline{\log 175 - \log 150 = x (\log 2375 - \log 2100)}$$

$$x = \frac{\log 175 - \log 150}{\log 2375 - \log 2100} = \frac{2.243038 - 2.176091}{3.375664 - 3.322219}$$

$$x = \frac{0.066947}{0.053445} = 1.2526335$$

$$x = 1.25 \text{ (approx.)}$$

$$\log 175 = \log K + 1.25 (\log 2375)$$

$$2.243038 = \log K + 1.25 (3.375664)$$

$$\log K = 2.243038 - 4.1219580 = -1.976542$$

$$\log K = 2.023458$$

$$K = 0.010555$$

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Problems containing numbers with fractional exponents such as a/b may be solved by using the exponent either in the given form (a/b), or by using its decimal equivalent. Since the exact decimal equivalent of many fractions can not be expressed, even though the division is carried to several places, the results obtained using the decimal form will be slightly less accurate than if the original fractional form had been retained.

Example:

$$y = \left(\frac{0.08726}{0.1321} \right)^{5/3}$$

$$\log y = 5/3 \log \left(\frac{0.08726}{0.1321} \right) = 5/3 (\log 0.08726 - \log 0.1321)$$

$$\log y = 5/3 (2.940815) - (1.120903)$$

$$= 5/3 (18.940815 - 20) - (9.120903 - 10)$$

$$= 5/3 (9.819912 - 10) = 5/3 (29.819912 - 30)$$

$$= 5 \left(\frac{29.819912 - 30}{3} \right) = 5 (9.939971 - 10)$$

$$= 49.699855 - 50 = -1.699855$$

$$y = 0.50101979$$

$$= .50102 \text{ (approx.)}$$

17
86

Logarithms are not applicable to the operations of addition or subtraction, yet in some equations the various terms, although added or subtracted, are in themselves, products, quotients, powers or roots. The solution of such equations may be written out algebraically by the insertion of the word "antilog" in the proper place as shown below. Or it may be advisable to remove such terms from the equation, compute its value by the use of logarithms, and then replace the given term by its numerical equivalent in the equation. The resulting equation can then be solved by ordinary algebraic methods.

$$E = \left[1 - \frac{1}{6.4} .408 \right] 100$$

$$\begin{aligned}\log E &= \log [1 - \text{antilog}.408 (-\log 6.4)] + \log 100 \\ \log E &= \log [1 - \text{antilog}.408 (-0.806180)] + \log 100 \\ \log E &= \log [1 - \text{antilog} (-.328921)] + \log 100 \\ \log E &= \log [1 - \text{antilog} 1.671079)] + \log 100 \\ \log E &= \log [1 - .4689] + \log 100 \\ \log E &= \log .5311 + \log 100 = 1.725176 + 2.000000 \\ \log E &= 1.725176 \\ E &= 53.11\end{aligned}$$

An alternate solution appears to be considerably simpler.

$$\begin{aligned}\text{let } Y &= \left(\frac{1}{6.4} \right) .408 \\ \log Y &= .408 (-\log 6.4) = .408 (-0.806180) \\ \log Y &= -.328921 = 1.617079 \\ Y &= .4689 \\ E &= (1 - .4689) 100 = (.5311) 100 = 53.11\end{aligned}$$

SOLUTION OF TRIANGLES USING LOGARITHMS

The arithmetical labor required in the solution of triangles is reduced if the multiplication and division of the sides and functions of the angles are replaced by the addition and subtraction of the logarithms of the numbers involved. The multiplication and division of numbers by the use of logarithms is described in the section entitled FUNDAMENTAL OPERATIONS WITH LOGARITHMS.

The logarithms of any trigonometric function of a given angle can be determined by first finding the value of the trigonometric function from a table of natural trigonometric functions, and then finding the logarithm of this number from an ordinary table of logarithms. The angle corresponding to a given logarithm of a trigonometric function can be determined by the reverse to this procedure.

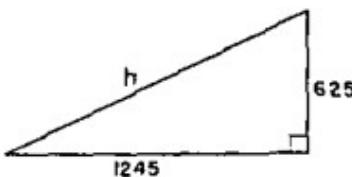
However, tables are available which give directly the value of the logarithms of the functions of all angles. These tables are designated as tables of logarithmic functions to differentiate them from tables of natural trigonometric functions. Whenever such a table is available it should be used, since the solution of the problem is greatly simplified.

The solution of triangles by logarithms involves the same principles and formulas that are used in solutions by natural functions. These formulas are simply the definitions of the trigonometric functions, sine, cosine, tangent, etc., in the case of right triangles, or the sine proportion and the tangent law when applied to oblique triangles. Since the cosine law is not expressed in terms of either ratios or products, it is not adapted to use with logarithms. For this reason the law of tangents has been included in the section entitled OBLIQUE TRIANGLES SOLVED BY SPECIAL FORMULAS. This formula is adapted to logarithmic computations and may be used to solve one of

SOLUTION OF EQUATIONS

the types of oblique triangles ordinarily solved by the cosine law, that is, when two sides and the included angle are given. The remaining case covered by the cosine law, three sides only, may be solved by logarithms by using a set of formulas known as the half-angles which are tabulated on page 169.

Logarithmic Solution of Right Triangles



$$b^2 = 625^2 + 1245^2$$

$$b^2 = \text{antilog} (2 \log 625) + \text{antilog} (2 \log 1245)$$

$$b^2 = \text{antilog} (2 \times 2.795880) + \text{antilog} (2 \times 3.095169)$$

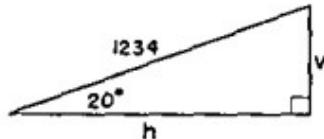
$$b^2 = \text{antilog } 5.591760 + \text{antilog } 6.190338$$

$$b^2 = 390,525 + 1,555,024 = 1,945,529$$

$$\log b^2 = 2 \log b = \log 1,945,529 = 6.289,038$$

$$\log b = 3.144,519$$

$$b = 1394.82$$



$$v = 1234 \sin 20^\circ$$

$$\log v = \log 1234 + \log \sin 20^\circ$$

$$\log v = 3.091315 + (9.534052 - 10)$$

$$\log v = 2.625367$$

$$v = 422.05$$

$$b = 1234 \cos 20^\circ$$

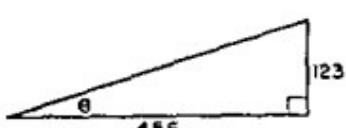
$$\log b = \log 1234 + \log \cos 20^\circ$$

$$\log b = 3.091315 + (9.972986 - 10)$$

$$\log b = 3.064301$$

$$b = 1159.58$$

In finding $\log \sin 20^\circ$ from a table of logarithmic functions, the value is found to be 9.534052. The (-10) is not annexed to the tables in favor of brevity, but must be used in the solution. That the characteristic of the $\log \sin 20^\circ$ is negative is evident from the fact that the natural sin of any angle is always less than unity. The use of the (-10) in conjunction with $\log \cos 20^\circ$ is governed by the same principle.



$$\tan \theta = \frac{123}{456}$$

$$\log \tan \theta = \log 123 - \log 456$$

$$\log \tan \theta = 2.089905 - 2.658965$$

$$\log \tan \theta = (12.089905 - 10) - 2.658965$$

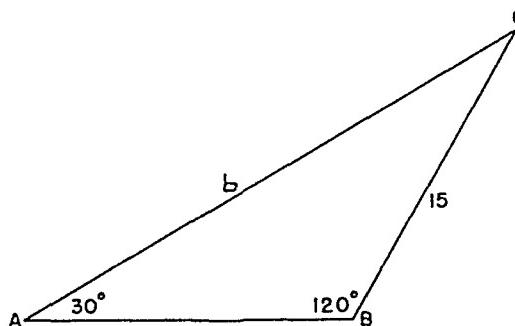
$$\log \tan \theta = 9.430940 - 10$$

$$\theta = 15^\circ 4.73'$$

Logarithmic Solution of Oblique Triangles

Two Angles and Any Side

(This is equivalent to having all three angles and any side given.)



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$b = \frac{(a) (\sin B)}{\sin A} = \frac{(15) (\sin 120^\circ)}{\sin 30^\circ}$$

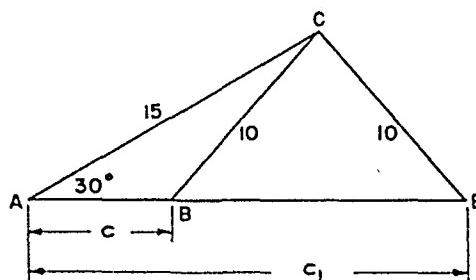
$$\log b = \log 15 + \log \sin 120^\circ - \log \sin 30^\circ$$

$$\log b = 1.176091 + (9.937531 - 10) - (9.698970 - 10) = 1.41465$$

$$b = 25.98 \text{ (approximately)}$$

$$C = 180^\circ - 30^\circ - 120^\circ = 30^\circ$$

$$c = 15$$

Two Sides and an Angle Opposite One of Them—Ambiguous Case

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{(b) (\sin A)}{a} = \frac{(15) (\sin 30^\circ)}{10}$$

$$\log \sin B = \log 15 + \log \sin 30^\circ - \log 10$$

$$\log \sin B = 1.176091 + (9.698970 - 10) - 1.00000$$

$$\log \sin B = 9.875061 - 1$$

$$B = 48^\circ 35.42' \text{ or } 180^\circ - 48^\circ 35.42' = 131^\circ 24.58'$$

$$C = 180^\circ - 30^\circ - 48^\circ 35.41' = 101^\circ 24.58' \text{ or, } \\ 180^\circ - 30^\circ - 131^\circ 24.58 = 18^\circ 35.42'$$

SOLUTION OF EQUATIONS

$$\frac{a}{\sin A} = \frac{c_1}{\sin 18^\circ 35.42'}$$

$$c_1 = \frac{(10) (\sin 18^\circ 35.42')}{\sin 30^\circ}$$

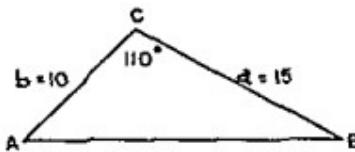
$$\begin{aligned}\log c_1 &= \log 10 + \log \sin 18^\circ 35.42' - \log \sin 30^\circ \\ \log c_1 &= 1.000000 + (9.503518 - 10) - (9.698970 - 10) \\ \log c_1 &= 0.804548 \\ c_1 &\approx 6.38\end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin 101^\circ 24 58'}$$

$$c = \frac{(10) (\sin 101^\circ 24 58')}{\sin 30^\circ}$$

$$\begin{aligned}\log c &= \log 10^\circ + \log \sin 101^\circ 24 58' - \log \sin 30^\circ \\ \log c &= 1.000000 + (9.991332 - 10) - (9.698970 - 10) \\ \log c &= 1.292362 \\ c &\approx 19.60\end{aligned}$$

Two Sides and the Included Angle



$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}$$

$$A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B) \quad (\text{Rule 227})$$

$$B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B) \quad (\text{Rule 228})$$

$$A + B = 180^\circ - 110^\circ = 70^\circ$$

$$\tan \frac{1}{2}(A - B) = \frac{\tan \frac{1}{2}(A + B)(a - b)}{(a + b)}$$

$$\tan \frac{1}{2}(A - B) = \frac{\tan \frac{1}{2}(70^\circ)(15 - 10)}{(15 + 10)} = \frac{(\tan 35^\circ)(15)}{25}$$

$$\begin{aligned}\log \tan \frac{1}{2}(A - B) &= \log \tan 35^\circ + \log 5 - \log 25 \\ &= (9.845277 - 10) - 0.698970 - 1.397940 = 9.146307 - 10\end{aligned}$$

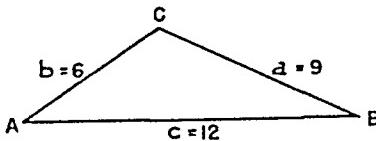
$$\begin{aligned}\frac{1}{2}(A - B) &= 7^\circ 58.37' \\ A - B &= 15^\circ 56.74'\end{aligned}$$

$$\tan \frac{1}{2}(A + B) = \frac{\tan \frac{1}{2}(A - B)(a + b)}{(a - b)}$$

$$\tan \frac{1}{2}(A + B) = \frac{\tan \frac{1}{2}(15^\circ 56.74')(15 + 10)}{(15 - 10)} = \frac{(\tan 7^\circ 58.37')(25)}{5}$$

$$\begin{aligned}\log \tan \frac{1}{2}(A+B) &= \log \tan 7^\circ 58.37' + \log 25 - \log 5 \\&= (9.146307 - 10) + 1.397940 - .698970 = 9.845277 - 10 \\ \frac{1}{2}(A+B) &= 35^\circ 0.19' \\ A+B &= 70^\circ 0.38' \\ A &= \frac{1}{2}(A+B) + \frac{1}{2}(A-B) = 35^\circ 0.19 + 7^\circ 58.37' = 42^\circ 58.56' \\ B &= \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = 35^\circ 0.19 - 7^\circ 58.37' = 27^\circ 1.82'\end{aligned}$$

Three Sides Given



$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad (\text{Rule 253})$$

$$\begin{aligned}s &= \frac{1}{2}(a+b+c) = \frac{1}{2}(9+6+12) = 13.5 \\(s-b) &= 13.5 - 6 = 7.5 \\(s-c) &= 13.5 - 12 = 1.5\end{aligned}$$

$$\sin \frac{A}{2} = \sqrt{\frac{7.5 \times 1.5}{6 \times 12}}$$

$$\begin{aligned}\log \sin \frac{A}{2} &= \frac{1}{2}(\log 7.5 + \log 1.5 - \log 6 - \log 12) \\&= \frac{1}{2}(0.875061 + 0.176091 - 0.778151 - 1.079181)\end{aligned}$$

$$\log \sin \frac{A}{2} = \frac{1}{2}(-0.806180) = -0.403090 = 1.596910$$

$$\frac{A}{2} = 23^\circ 17.02'$$

$$A = 46^\circ 34.04'$$

NATURAL OR NAPERIAN LOGARITHMS

Although logarithms to the base 10 are most commonly employed, there are several instances in which another base is used. This base is an irrational number which for all practical purposes is taken as 2.718. For simplicity, this value is referred to as (*e*), and logarithms to this base are called logarithms to the base (*e*) (written log_{*e*}). These logarithms are also designated as either natural or Napierian logarithms to distinguish them from common logarithms.

Tables of natural logarithms which are used as described below are available. However, the log_{*e*} of any number can be found using a table of common logarithms together with the appropriate conversion factors. These factors are irrational numbers 2.302585... and its reciprocal 0.434294.... For simplicity, these are assumed to be 2.3026 and 0.4343 respectively.

SOLUTION OF EQUATIONS

$$\log_e A = 2.3026 \log_{10} A \quad (\text{where } A \text{ is any number})$$

$$\log_e A = 0.4343 \log_{10} A$$

It is apparent that the natural logarithm of any number is approximately equal to 2.3026 times the common logarithm of that number, and conversely, the common logarithm is 1/2.3026 or 0.4343 times the natural logarithm. The natural logarithms of the numbers 1 to 10 are listed in all natural logarithm tables. Some tables include fractional numbers (0.1 to 1.0) as well. The method of obtaining natural logarithms as now described assumes the use of the less extensive table. For the logarithms of numbers 1 to 10 the values are given directly in the tables. Thus

$$\begin{aligned}\log_e 2 &= 0.6931 \\ \log_e 4 &= 1.3863 \\ \log_e 8 &= 2.0794\end{aligned}$$

Where interpolation is required, the same procedure is employed as for common logarithms.

To obtain \log_e of any number less than 1.0 or greater than 10 it should be noted that when the decimal point of the number is moved one place to the right the natural logarithm of the resulting number can be obtained by adding 2.3026 to the natural logarithm of the original number. Likewise 2.3026 should be subtracted when the decimal point is moved one place to the left. Moving the decimal point (n) places in either direction requires the addition or subtraction of n times 2.3026 to the natural logarithm of the original number. These facts may be used to formulate rules for finding the natural logarithms of numbers not given directly in the tables.

- Step 1. Move the decimal point in the original number so that the number produced has an absolute value between 1 and 10.
- Step 2. Find the natural logarithm of the number obtained in step 1.
- Step 3. If the decimal point in step 1 was moved n places to the left, the logarithm obtained in step 2 should be increased by the product of $n(2.3026)$.

$$\begin{aligned}\log_e 125 &= \log_e 1.25 + 2(2.3026) \\ &= 0.2231 + 4.6052 = 4.8283\end{aligned}$$

If the decimal point in step 1 was moved n places to the right, the logarithm obtained in step 2 should be decreased by the product of $n(2.3026)$.

Section V

ANALYTICAL GEOMETRY OF STRAIGHT LINES

INTRODUCTION

The chief feature of analytical geometry, which distinguishes it from ordinary plane geometry, is the extensive use of algebraic and trigonometric methods. The explanation of the principles of this subject is made by the use of graphs or curves of the various equations. For simplicity, and because they are of most practical importance, the analytical geometry of straight lines only is considered in this section.

To graphically represent an equation involving two variables, it is necessary to establish a number of points each of which will have coordinates (x) and (y) which will satisfy the given equation. The numerical values for the coordinates of any one point are found by assigning some arbitrary numerical value to one of the variables and then solving the equation for the corresponding value of the other variable. A number of such points can be similarly established and through these a continuous curve or line can be drawn to represent the equation.

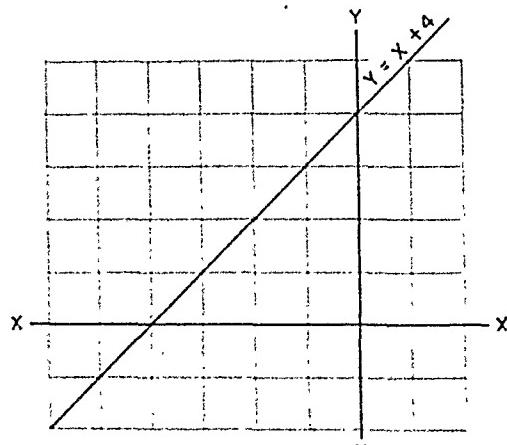
The variable to which the arbitrary value is assigned is called the *independent variable*, and the remaining unknown, since it is a function of the independent variable, becomes the *dependent variable*. The value of the independent variable is customarily plotted as abscissa (x) and the value of the dependent variable as ordinate (y). However, this order may be reversed at any time, even when plotting successive points in the same equation.

The plotting of a typical equation is shown by the following example.

$$y = x + 4$$

This equation may be rearranged in any manner which may simplify the finding of pairs of value for the coordinates (x) and (y), but it is entirely workable in its given form. The values of various pairs of the independent and dependent variables are found and tabulated as shown.

LET $X =$	THEN $Y =$
0	4
2	6
-2	2
-4	0
-6	-2

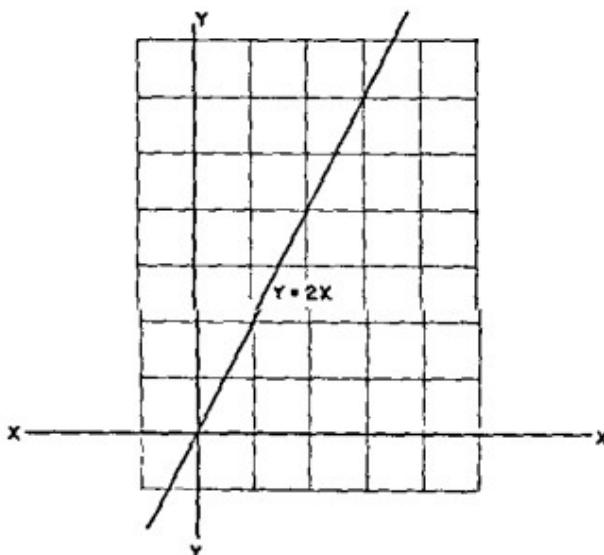


These points, when plotted indicate that the given equation might represent a straight line. Inasmuch as each of the several values assigned to the independent variable was chosen at random, it seems unnecessary to plot additional points, but as an expedient to draw a straight line through those points already established. This line, called the curve of the equation, will indicate at a glance what value of (y) corresponds to any given value of (x), or vice-versa.

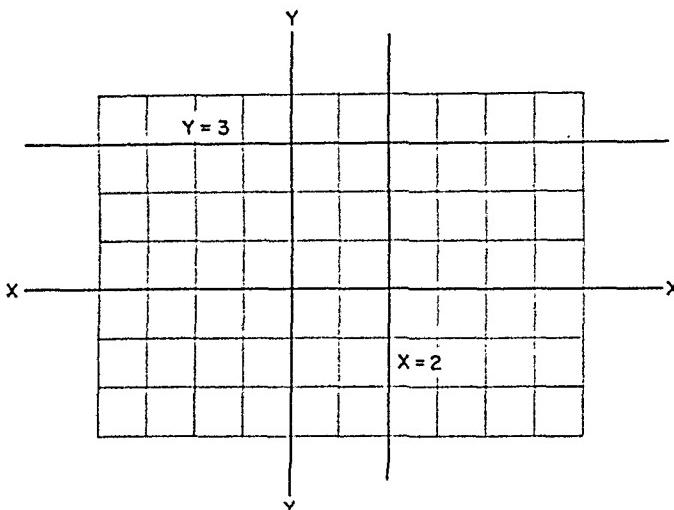
STRAIGHT LINES

An equation of the general form $Ax \pm By \pm C = 0$, is called a linear equation since all its plotted points will fall on a straight line. This will always be the case where both of the variables appear separately, that is, not as their product, xy , or their quotient, x/y , and with exponents equal to one. Since a straight line is determined by knowing two points on it, the graph of a linear equation can be drawn when only two points have been plotted. Usually, but not necessarily always, the most convenient points to choose are those located where the line crosses the two axes. These two points are found by assuming $x = 0$ and finding (y), and then letting $y = 0$ and finding (x). The two values thus found for (x) and (y) are called *intercepts* as the line crosses the axes at these points.

In some cases the X -intercept and the Y -intercept are both zero and consequently the line passes through the origin. This condition is at once apparent whenever the value of the dependent variable is zero for a corresponding zero value of the independent variable. Straight lines which pass through the origin can be plotted whenever one other pair of values of (x) and (y) are known.

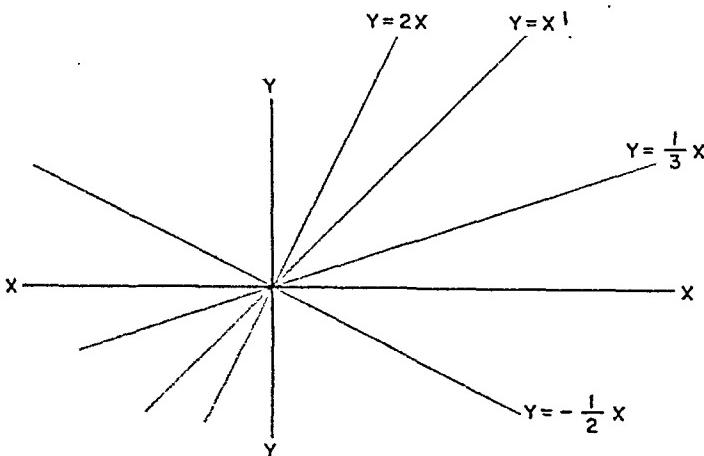


In some cases only one variable is represented in the equation, as for example $x = 2$, $y = 3$, etc. In these examples, the curve of the equation is a straight line parallel to one of the reference axes. When $x = a$, the graph is a straight line parallel to the $Y-Y$ axis; and $y = b$ is a straight line parallel to the $X-X$ axis.



Slope of a Straight Line

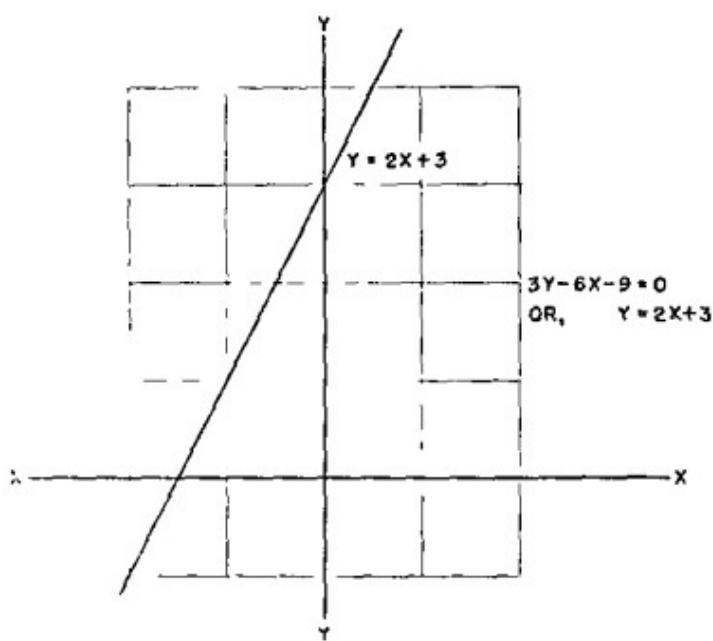
As a point moves along a straight line, any change in (x) produces a definite change in (y). In the equation $y = x$ it is apparent that for any change in (x) the value of (y) will be equally affected. This ratio of the *change* in (y) to the *change* in (x) is called the slope of the curve, and for the equation $y = x$ its numerical value is equal to one. In the equation $y = 2x$ any change in (x) causes twice as large a change in (y), that is, it has a slope of two. The slope of a line is considered positive when it ascends from left to right; and a line which descends from left to right has a negative slope. A line parallel to the X - X axis has a zero slope, and one perpendicular to the X - X axis is said to have no slope.



An inspection of the lines so far plotted show that where the slope is positive, (y) increases as (x) increases, and where the slope is negative, (y) decreases as (x) increases. Furthermore, where the slope is more than one, the value of (y) changes at a greater rate than (x); where the slope is less than one, the value of (y) changes at a

SOLUTION OF EQUATIONS

rate smaller than (x). It is apparent that the slope term is useful in indicating the direction in which the curve of the equation will exist when plotted. This slope term will always be the coefficient of the (x) term when the given equation is solved for (y). For example, the slope of the equation $3y - 6x - 9 = 0$, is 2 since this is the coefficient of the (x) term when the equation is solved for (y), as shown below.

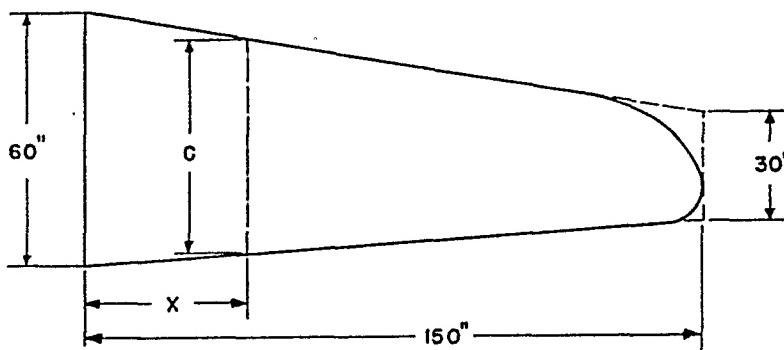


} Any constant term, such as 3 in the above example, which remains after the equation is solved for (y), will signify that the line does not pass through the origin. Instead, the line crosses the Y - Y axis at the ordinate corresponding to the numerical value of the constant term (customarily represented by b). The general equation for a straight line is now written:

$$(263). \quad y = mx + b$$

In this equation, (m) and (b) are the slope and Y -intercept respectively. These quantities may be integers or fractions, and of either plus or minus value. This form of the equation for a straight line is known as the slope-intercept form, since it indicates at a glance what the slope and Y -intercept of any given equation might be. If in any case there is no constant, or (b) term, then the equation becomes $y = mx$, and as shown in Fig. 0 the line passes through the origin.

There are many practical applications of the straight line equation when written in the slope-intercept form, especially in problems relating to motion, energy, lead diagrams, etc. The following example demonstrates its use in finding the chord length of an airplane wing when the chord length varies inversely as to its distance from the center of the airplane.



$$c = 60 - \left[\frac{60 - 30}{150} \right] x = 60 - .2x$$

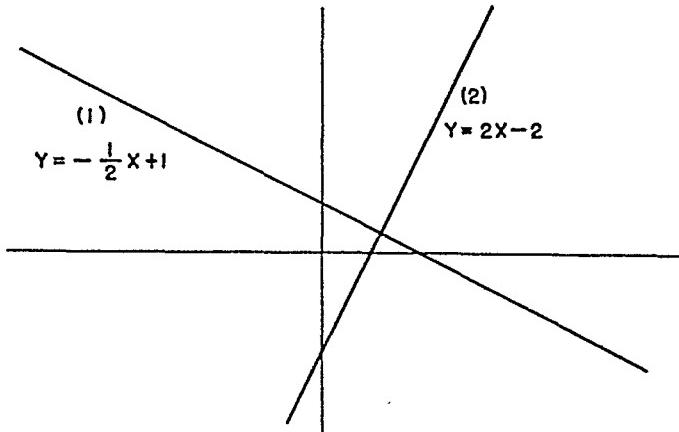
Interpretation of Slope Term

As previously stated, a positive slope indicates that y increases with x , and a negative slope indicates that y decreases with x . Where (m) is greater than one, the value of (y) changes at a greater rate than (x). Where (m) is less than unity the value of (y) changes at a smaller rate than (x).

It is apparent that equations having equal slopes will plot as parallel lines. Two lines which intersect at right angles may be identified without plotting by noting that the slope of one is the negative reciprocal of the other. One number is the negative reciprocal of another if their product is -1 . Thus, the negative reciprocal of 3 is $-1/3$; of $-2/5$ is $5/2$; etc. Obviously the reciprocal of a fraction is merely the fraction inverted.

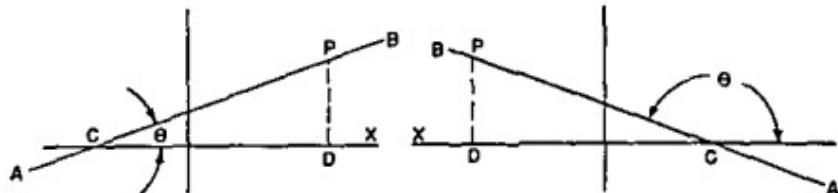
$$(A) \quad 2x + 4y = 4 \\ y = -\frac{1}{2}(x) + 1$$

$$(B) \quad -2x + y = -2 \\ y = 2x - 2$$



The slope term (m) of a straight line may be used to advantage in engineering problems. So far the slope has been defined as the ratio of the change of ordinate (y)

to the change in abscissa (x) between any two points on the line. The slope of a straight line may also be defined as the tangent of the angle between the axis and the given line. The straight line AB cuts the X - X axis at point C . There are altogether four angles formed, but θ is the only angle commonly considered, it being the angle between the initial CX and the terminal side CB . At any point on the terminal side such as P , a line is drawn perpendicular to the X - X axis.



From trigonometry,

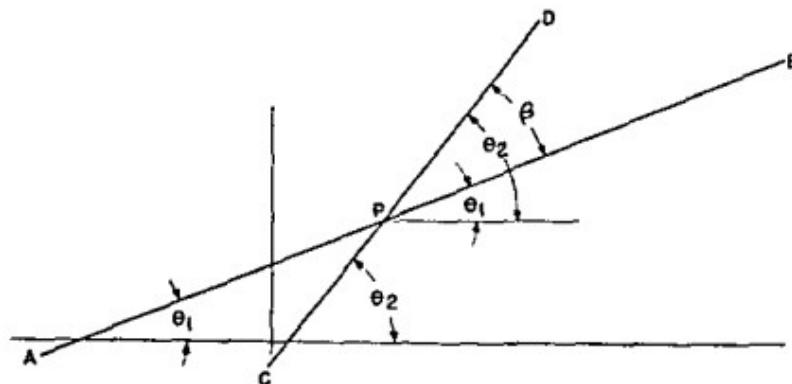
$$\tan \theta = PD/CD$$

But PD/CD is the slope of the line by fundamental definition, since it is the ratio of the change in ordinate to the change in abscissa between points C and P .

Therefore,

$$\tan \theta = m.$$

Where two straight lines intersect, and the angle between them is desired, it is possible to arrive at a solution by first finding the difference in slopes of the two lines.



Let straight lines AB and CD intersect at point P . The angle formed is β or,

$$\beta = \theta_2 - \theta_1$$

Therefore $\tan \beta = \tan (\theta_2 - \theta_1)$

From trigonometry, (Rule 234)

$$\tan (\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}$$

But $\tan \theta_2 = m_2$, and $\tan \theta_1 = m_1$ where m_1 is the slope of AB .

(264).

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If θ_2 is always taken greater than θ_1 , then $\tan \beta$ will be positive when β is less than 90° and negative when β is greater than 90° .

Example: Find the angle between the two straight lines whose equations are

$$(A) \quad x + 2y + 2 = 0$$

$$(B) \quad 2x - 3y + 5 = 0$$

Solution:

$$(A) \quad y = -\frac{x}{2} - 1$$

$$m = -\frac{1}{2}$$

$$(B) \quad y = \frac{2}{3}x + \frac{5}{3}$$

$$m = \frac{2}{3}$$

Since the first line makes the larger angle with the X axis, its slope will be designated as m_2 .

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-\frac{1}{2} - \frac{2}{3}}{1 - \frac{1}{2} \cdot \frac{2}{3}} = -\frac{7}{4}$$

Angle β is here an obtuse angle since its tangent is negative. The supplementary acute angle has a tangent of $+7/4$, and the angle itself is readily found from a table of trigonometric functions.

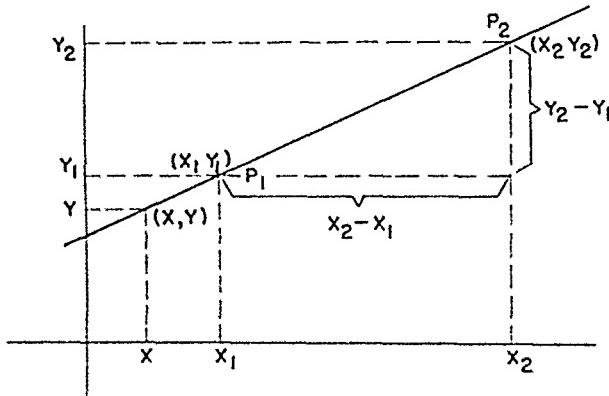
Equation of any Straight Line

The process of establishing an equation from certain known data is now to be described.

If the slope of the Y-intercept of a straight line are known, its equation can be obtained by substituting these values for (m) and (b) in the slope-intercept form of the straight line equation,

$$y = mx + b$$

If two points through which the line passes are known, the equation of the line can be obtained by an application of the fundamental definition of slope.



The points P_1 and P_2 are located by the coordinates (x_1, y_1) and (x_2, y_2) , respectively. The change in the value of the ordinates between these two points is $y_2 - y_1$ and the change in the value of the abscissa is $x_2 - x_1$. The ratio of these two changes is the fundamental definition of the slope of a straight line. Therefore, the slope of the line through these two points is:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Another point having coordinates (x, y) is selected somewhere on the line. The slope at this point is the same as at P_1 or P_2 , since a straight line has the same slope throughout its length. Therefore, the ratio

$$\frac{y_1 - y}{x_1 - x}, \text{ must be equal to the ratio}$$

$$\frac{y_2 - y_1}{x_2 - x_1}, \text{ or,}$$

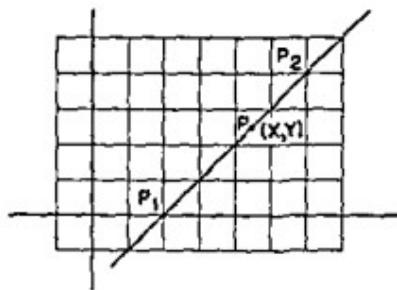
$$\frac{y_1 - y}{x_1 - x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(265).

This is known as the *two point form* of the equation of a straight line. It may be applied as follows

Example

Given, $P_1(2,0)$ and $P_2(6,4)$. Assume some point $P(x,y)$ which lies on a straight line, and then substitute in the above equation.



$$\frac{0 - y}{2 - x} = \frac{4 - 0}{6 - 2}$$

$$8 - 4x = -4y, \quad \text{or } x - y = 2$$

If one point through which the line passes and the slope are known, the point slope form of the equation of a straight line may be written. Let (x_1, y_1) be the given point and (m) the slope of the line. Another point having coordinates (x, y) is selected somewhere on the line. The slope of the line is then:

$$\frac{y - y_1}{x - x_1}$$

But (m) is also the slope, therefore,

$$\frac{y - y_1}{x - x_1} = m$$

Example

Given, $P_1 (4,2)$ and $m = 2/3$

$$\frac{y - 2}{x - 4} = 2/3$$

$$\begin{aligned} 2x - 8 &= 3y - 6 \\ 2x - 3y - 2 &= 0 \end{aligned}$$

Simultaneous Equations of Straight Lines

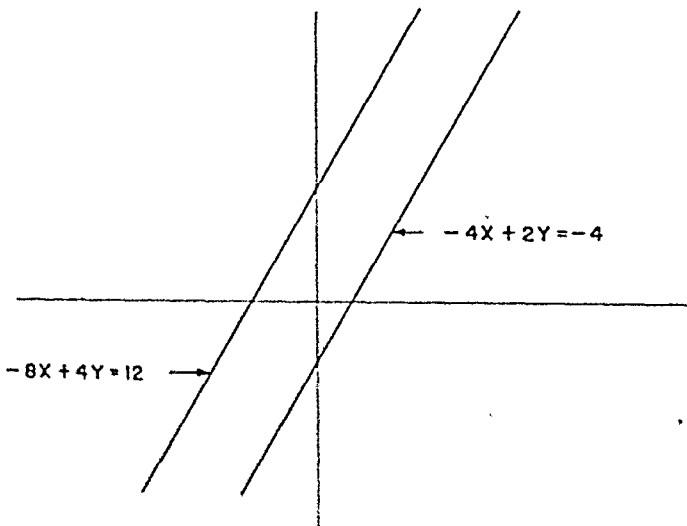
To solve a system of two or more simultaneous equations, it is necessary to obtain a set of values for the variables which will satisfy each of the equations. If the given equations represent straight lines, only one pair of such values can be found, and the equations are termed *independent*. Such equations can be solved graphically or by one or more of the several algebraic methods. However, not every system of equations has a common solution, and it is impossible to find any values for the variables which will satisfy each of the equations. Such equations are called *inconsistent*. In another instance a set of equations may be such that *any* values which will satisfy one of the equations will also satisfy the others. Equations of this nature are called *equivalent* or *same* equations, because each can be derived from the other.

Whether a system of equations are independent, inconsistent, or equivalent can be determined by graphical means. An alternate procedure is to change the equations to the slope intercept form and observe the nature of their slopes and intercepts.

To fulfill the requirements of *independent* equations, the graph of two straight lines must intersect at just one point. The coordinates of this point are the values of the variables which will satisfy the two given equations. It is apparent, then, that for a solution the lines must intersect, and therefore must not be parallel. In other words, they must be of unequal slope.

The graph of two equations which have the same slope are as shown.

$$\begin{aligned} -4x + 2y &= -4 \\ -8x + 4y &= 12 \end{aligned}$$



Since parallel lines have no points in common, there can be no pair of values satisfying both equations. Such equations are inconsistent and cannot be solved. Equations of this type are characterized by *equal slope terms and unequal Y-intercepts*.

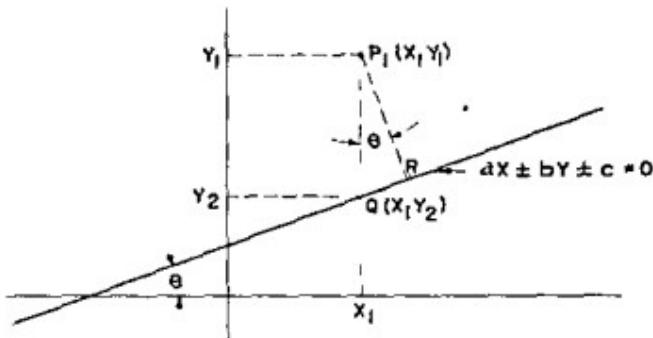
$$\begin{array}{l} -4x + 2y = -4 \quad \text{or} \quad y = 2x - 2 \\ -8x + 4y = 12 \quad \text{or} \quad y = 2x + 3 \end{array}$$

That the *Y-intercepts* must not be identical is apparent, for if parallel lines have the same intercepts the lines are coincident, and all points that lie in one of these lines must also lie in the other. Such equations are *equivalent* and cannot be solved for definite values of the unknowns. Two equations of this type are *identical* when each is reduced to its simplest form, or to the slope intercept form

$$\begin{array}{l} x + 2y = 2 \quad \text{or} \quad y = -x/2 + 1 \\ 3x + 6 = 6 \text{ or } x + 2y = 2 \text{ or, } y = -x/2 + 1 \end{array}$$

Distance From a Point to a Straight Line

The minimum distance from a point to a straight line is the length of the line drawn perpendicular to the line and through the given point



Through point $P_1(x_1, y_1)$ draw a perpendicular to the given straight line $ax \pm by \pm c = 0$, intersecting the line at R . An ordinate line to P_1 intersects the straight line at Q . The ordinate of point Q is y_2 and its abscissa is $x_2 = x_1$ since P_1 and Q are both on the same ordinate line. Point Q is on line $ax + by + c = 0$, and its coordinates can therefore be substituted in the equation.

$$ax_1 + by_1 + c = 0$$

$$\text{or} \qquad y_2 = -\frac{ax_1 + c}{b}$$

$$\text{Then } P_1Q = y_1 - y_2 = y_1 - \left[-\frac{ax_1 + c}{b} \right]$$

$$P_1Q = \frac{by_1}{b} + \frac{ax_1 + c}{b} = \frac{ax_1 + by_1 + c}{b}$$

It is evident that the distance P_1Q will be expressed as a positive quantity when $P_1(x_1, y_1)$ lies above the given line, and negative when below. From trigonometry,

the length of the perpendicular P_1R is equal to $P_1Q \cos \theta$. The tangent, and then the cosine, of θ is obtained from the equation of the straight line: By transposing it to the slope intercept form,

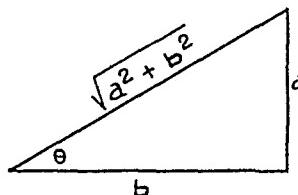
$$y = -\frac{a}{b}x - \frac{c}{b}$$

$$\tan \theta = m = -\frac{a}{b}$$

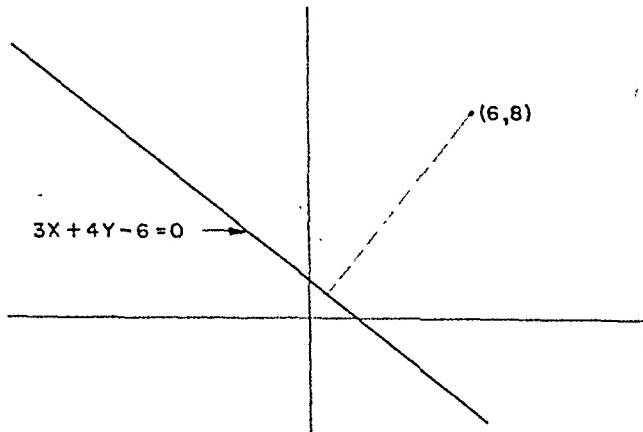
$$\text{Then } \cos \theta = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{From which, } P_1R = P_1Q \cos \theta = \left[\frac{ax_1 + by_1 + c}{b} \right] \left[\frac{b}{\sqrt{a^2 + b^2}} \right]$$

$$(266). \quad P_1R = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$



Example:



$$P_1R = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$a = 3, \\ x = 6,$$

$$b = 4, \\ y = 8$$

$$c = -6$$

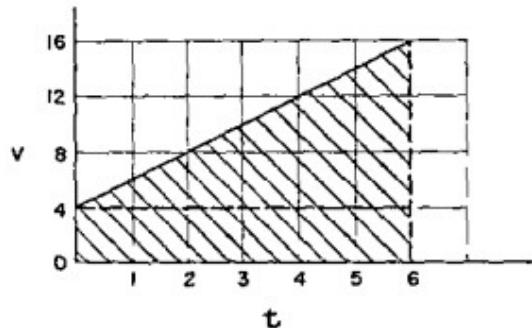
$$P_1R = \frac{(3)(6) + (4)(8) - 6}{\sqrt{9 + 16}} = \frac{44}{5} = 8.8$$

The perpendicularly distance between any two parallel lines can be found by the same method as for a point and a line. The coordinates of some point P contained in either of the lines is first determined. This is accomplished by assuming some value as abscissa, or ordinate, and then finding the corresponding value of the other coordinate. These values are then substituted in the point-line formula.

Area Beneath a Straight Line Segment

The area beneath a straight line between two points or limits and the X-X axis can be found by a simple computation since the bounded area will be either a rectangle, a triangle, or a combination of the two which is a trapezoid. A problem in uniformly accelerated rectilinear motion demonstrates this point.

A body in motion is being accelerated at a constant rate. In 6 seconds time its velocity changes from 4 ft/sec. to 16 ft/sec.



$$\text{Acceleration } a = \frac{16 - 4}{6} = 2 \text{ ft/sec.}^2$$

$$\begin{aligned}\text{Distance traversed } S &= \frac{1}{2}(16+4)(6) = 60 \text{ ft} \\ &= \text{Area beneath straight line.} \\ &= (4)(6) + \frac{1}{2}(16-4)(6) \\ &= 24 + 36 = 60 \text{ ft.}\end{aligned}$$

If the value of the average ordinate (average velocity) is desired, it is only necessary to average the ordinate at the first and last point. The average ordinate is also the ordinate at the middle of the graph under consideration.

The velocity at any point is represented by the ordinate of the straight-line at the time in question. To determine its value at any time (t) an equation is written by substituting the values of slope and Y intercept in the slope-intercept form of the straight-line.

$$V = 4 + \left[\frac{16-4}{6} \right] t = 4 + 2t \text{ or } 2t + 4$$

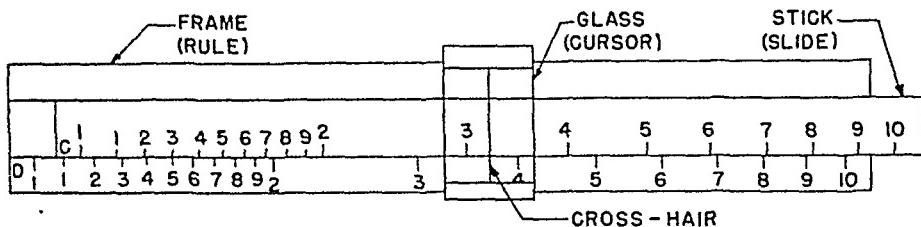
Section VI

APPENDIX — SLIDE RULE

Introduction

The slide rule is an instrument which saves time and labor in performing mathematical computations. The accuracy of the results obtained depend upon the size of the rule employed and the magnitude of the numbers involved. For most practical purposes the inaccuracies involved are not large enough to be of any consequence. For example, the ten-inch rule which is customarily used gives results correct to within one part in one thousand, or one-tenth of one per cent. Greater accuracy can be obtained by using larger rules whenever such precision is required.

A variety of slide rules have been developed which differ in size, arrangement, and also in the method of graduating the scales. However, all rules are used in a similar manner, and the description and method of operation described below for a normal ten-inch rule can be considered to apply, in a general way, to slide rules of all types.



The ordinary slide rule consists of three parts which for convenience will be called the *frame*, the *stick*, and the *glass*. These are more properly named the rule, the slide, and the cursor, respectively, although this terminology does not as readily suggest the part referred to as the names first given. Both the frame and the stick are graduated with various scales of which the *C* and *D* scales are the most used. These two scales are identical and lie adjacent to each other when the stick is installed in the frame with the front of the stick facing forward. The letters *C* and *D* do not always accompany their respective scales, but if employed they may be found at the left-hand edge of the rule and on a line with the scale which they identify.

The entire length of the *C* and *D* scales are graduated into *major* divisions which are numbered from 1 to 10. The extreme right-hand figure may be 1 instead of 10 on some rules. The left-hand digit 1, and the right-hand figure 1 or 10 of the scales are called the *left index* and the *right index*, respectively, of the scales on which they appear.

An inspection of the rule reveals that the distance between each two consecutively numbered major divisions is unequal, being largest for the 1 — 2 division, and decreasing progressively for the divisions to the right. Each of these numbered divisions are subdivided into ten *minor* divisions which are not numbered except for those graduations within the first major division. The length of each minor division, between

any two numbers of the major divisions, varies in the same way as the length of the major divisions vary along the scale.

The figures used in numbering the minor graduations within the first major division are slightly smaller than those used for the major divisions. They are also conspicuous by being abbreviated as only the second digit of the appropriate figure is represented. Thus, 12 appears as 2, 18 as 8, etc. To avoid confusion in reading these numbers, it is recommended that the omitted digit 1 be temporarily marked on the scales until experience justifies its omission. The abbreviation of numbers in some parts, and the lack of any numbering system whatsoever for the remaining minor graduations of the rule, is made necessary because of the limited space available. For the same reason it is necessary to vary the number, and consequently the value of the smaller graduations which appear between the minor graduations of the rule. For example, on the ordinary ten-inch slide rule, the minor divisions within the first major division (1—2) are each subdivided into ten parts. Between the major divisions (2—3) and (3—4) these spaces are divided into five parts. The minor divisions throughout the remainder of the scale are divided into two parts only. The value to be assigned to any one of these smallest divisions is as defined below.

The position of any number on the C and D scales of the slide rule is governed by the sequence of digits making up that number, and the position of the decimal point is of no consequence until after the work of the rule has been completed. Thus, the numbers 25, 25 and 0.25 all occupy the same position. Similarly, the value of the 1 mark on the left-hand of the scale may represent 1, 10, 100, 0.01, etc. If, for example, 10 is the value assumed for the left index, then the values for the major graduation numbers throughout the scale becomes 20, 30, 40, etc. The numbered minor divisions between 10 and 20 will be read as 11, 12, 13, etc., since in this case the value of each of these divisions is equal to 1. Each of these numbered divisions (11, 12, 13, etc.) are subdivided into ten spaces, consequently the value of each subdivision is 0.1. Assuming that the left index is still considered to be 10, the smallest graduations between 20 and 30 and between 30 and 40 will each represent 0.2 since there are five of these graduations to each minor division which have a value of 1. Between 40 and the right end of the scale the smallest graduations have a value of 0.5 each.

After a small amount of experience is gained in the use of the slide rule, the variation in the value of the smallest divisions appearing on the scales will cause but little difficulty. Whenever doubt exists as to the value to be assigned to any graduation of the slide rule, it is advisable to count between two known points on the scale, one value being less than the unknown value and the other greater. In this way the true value of each division is quickly obtained, and from this, the true value of the position in question is determined.

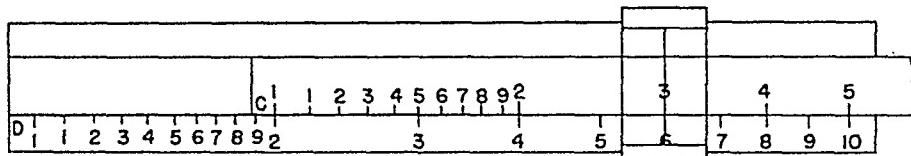
Before undertaking the explanation of how the slide rule may be employed in performing divisions, multiplications, or obtaining square roots of numbers, it is important to have fixed firmly in mind that the results obtained from any of these three operations will always be found on the D scale. No time and effort need be wasted in searching for answers if this fact is remembered. The operation of the slide rule is most easily learned by employing simple examples for which the answers are known. More complex problems can afterwards be tried for practice to obtain accuracy and speed.

FUNDAMENTAL OPERATIONS WITH A SLIDE RULE

Division

The operation of division is the simplest of all computations which can be performed on the slide rule. For the purpose of explanation, the dividend will be referred to as the numerator, and the divisor will be called the denominator. This is a logical procedure for whenever any division is indicated by writing the terms in fractional form, the dividend and the divisor are placed in these respective positions.

In performing divisions, the identical *C* and *D* scales are used. The numerator of the fraction is first set with the glass on the *D* scale. Then the denominator on the *C* scale is set directly above the numerator on the *D* scale, and the quotient is found on the *D* scale at a point directly beneath either the left or the right index of the *C* scale. Thus, to divide 6 by 3 the number 6 is first set on the *D* scale. This is most easily accomplished by moving the glass along the rule until the hair line of the glass coincides with the 6. Then the stick is moved to either the left or right until the 3 of the *C* scale is directly above the 6 on the *D* scale. In this position the two numbers are both in line with the hair line of the glass. The quotient, 2, is then read on the *D* scale at a point directly beneath the left index of the *C* scale.



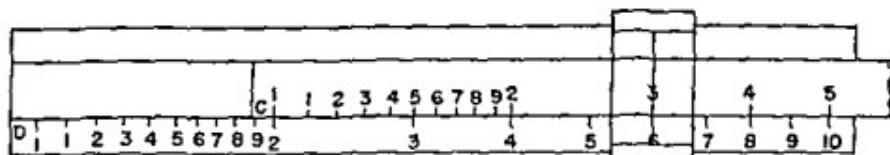
It is helpful to think of the procedure described above as establishing the fraction on the *C* and *D* scales so that the position of the numerator and denominator are interchanged with their normal positions when the desired division is written in fractional form. That is, the numerator is on the *D* scale with the denominator directly above it on the *C* scale.

The position of the decimal point in the result obtained can usually be placed by inspection. Where this is not possible, a rough arithmetical computation will serve to properly locate its position. (Also see POSITION OF THE DECIMAL POINT DETERMINED BY CHARACTERISTICS. Page 213.)

Multiplication

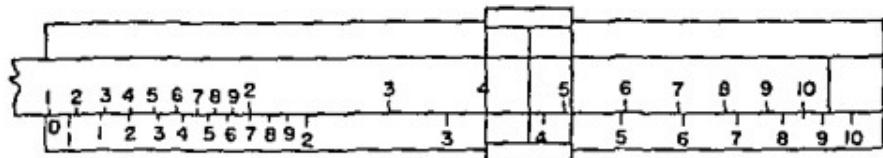
The operation of multiplication is a reverse process to that of division, and consequently the same two scales (*C* and *D*) are employed. For the purpose of explanation it will be assumed that only two factors are being multiplied, for regardless of the actual number, the product of any number of factors is always obtained by multiplying together only two numbers at a time.

To obtain the product of two numbers, set the index, 1, of the *C* scale opposite one of the factors on the *D* scale. Then move the glass along the rule until the second factor is located on the *C* scale. The product is then found directly below on the *D* scale.



Thus, to multiply 2 by 3, set the left index of the *C* scale directly above the 2 on the *D* scale. Then move the glass to the right along the rule until the 3 is located on the *C* scale. The product, which is 6, is then found directly below on the *D* scale.

If the product of the first two digits being multiplied together exceeds 10, it will be found that the second factor, which is to be located on the *C* scale, will project beyond the frame to the right. In this case the stick must be reversed, that is, extended in the opposite direction, so that the right index instead of the left is used. With the right index placed directly above one of the factors on the *D* scale, the glass is moved to the left until the second factor is located on the *C* scale. The product is then found directly below on the *D* scale.



Thus, to multiply 45 by 85, set the *right* index of the *C* scale directly above 85 on the *D* scale. Then move the glass to the left until the 45 is located on the *C* scale. The product, which is 3825, is then found directly below on the *D* scale. Whether to use the left or the right index of the *C* scale when performing multiplications is of no great importance, as the fact is soon discovered when the opposite index must be employed. However, to effect a saving in time and labor, the following rule will be found useful in determining which index to use.

If the product of the first figures of each of the given numbers is less than 10, use the left index; if this product is greater than 10, use the right index.

An exception to this rule will be found in such cases as 3.12 times 3.31. According to the rule the left index should be used. It will be found, however, that it is necessary to use the right index. This is due to the fact that although the product of the first digits of the two numbers is less than 10, the product of the complete numbers is greater than 10.

As seen from the examples given, the rule is not without exception. However, in most cases the rule applies, and its consistent use will save an appreciable amount of time and effort if numerous computations are to be made.

Combined Multiplication and Division

For continued operations involving both multiplication and division such as

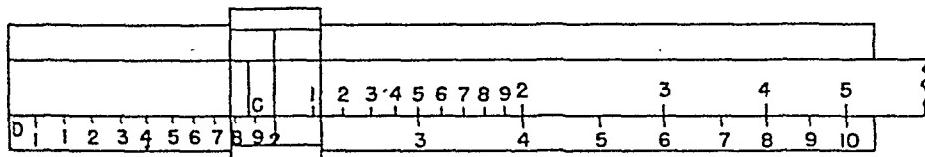
$$N = \frac{(a)(b)(c)}{(x)(y)(z)}$$

proceed by dividing the first factor in the numerator by the first divisor in the denominator. Next multiply the quotient obtained by the second factor in the numerator and then divide by the second factor in the denominator, etc., thus progressing through the problem according to a saw-tooth plan. Solving the problem in this manner often eliminates several settings and readings of the slide rule, thereby lessening the labor involved and diminishing the possibilities of errors.

In the explanation above, it was stated that the first factor of the numerator was to be divided by the first divisor of the denominator, etc. However, the order in which these terms are used is unimportant provided that the numbers are chosen alternately from the numerator and the denominator. Experience with such problems will reveal that the order in which the values are given is not always the most practical order in which the problem can be solved. Another fact of importance is that if there are more factors in the numerator than there are divisors in the denominator, or vice versa, it is possible to use 1 as a factor or divisor as many times as may be necessary.

Proportions

If the left index of the C scale is placed directly above the number 2 of the D scale, the value of the ratio (fraction) established will be $1/2$. Other ratios having the same value will be found to exist at all adjacent points along the C and D scales.



Other arbitrary positions of the two scales will verify the fact that for any position of the stick, the numbers of the C scale bear a constant ratio to those on the D scale. Thus, to solve a proportion such as

$$\frac{x}{62.4} = \frac{231}{1728} \quad \text{which is the same as} \quad \frac{231}{1728} = \frac{x}{62.4}$$

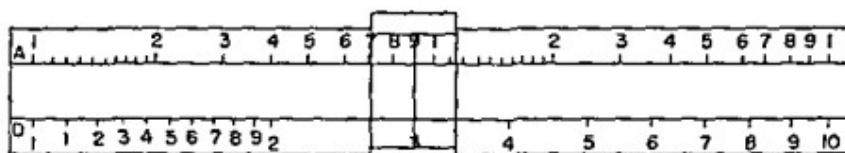
it is only necessary to place 231 on the C scale directly above 1728 of the D scale. Then look along the D scale for 62.4. The numerical value of (x) will be found directly above the 62.4. In this case the value of (x) is 8.34.

Other problems in proportion are similarly solved. This method of procedure is of advantage in that the operation of cross-multiplying is eliminated, thus saving time, labor and possibility of error.

Square Roots and Squares of Numbers

The square root of a number is a quantity which when multiplied by itself, will produce the original number. Thus, $3 = \sqrt{9}$, since $(3)(3) = 9$. The operation of finding the square root of a number involves the use of the *A* and the *D* scales. Since neither of these scales are found on the stick, it may be removed from the rule when obtaining square root, thus eliminating a possible source of confusion. The *A* scale can be quickly identified on the rule since it consists of two identically graduated scales, placed consecutively, at the top of the frame. The total length of these two scales is the same as the full length of one *C* or *D* scale and each is graduated in a similar manner, although the subdivisions are fewer in number and consequently have a larger value. In many cases, the *A* scale is on the front of the rule so that the four visible scales are the *A*, *B*, *C*, and *D*, reading from the top to the bottom. However, on some rules the *A* scale is on the side opposite to that of the *C* and *D* scales. In these cases, the *D* scale of the rule is repeated on the back side of the frame so that in all cases there is a *D* scale directly below the *A* scale.

Each half of the *A* scale is graduated in a manner similar to the *D* scale, but since the former is only one-half as long, the graduations are fewer and consequently have a larger value. The position of any number can be established as easily on the *A* scale as on the *C* or *D* scales if the method of counting as described on page 205 is employed.



The square root of any number located on the *A* scale is found at a point on the *D* scale directly below that number. For locating this point on the *D* scale the cross-hair of the glass is invariably employed. Since the *A* scale consists of two equal and identical sections, it becomes necessary to determine on which one of the two scales the given number should be located. Of the several rules available for governing this decision, the most infallible seems to be to count the number of digits occurring to the left of the decimal point in the given number. If this *number* of digits is odd, such as 1, 3, 5, etc., the left hand section is used, if the number of digits is even, such as 2, 4, 6, etc., the right-hand section is used.

The position of the decimal point in the answer read from the *D* scale is usually apparent and no question is involved in its determination. However, it may be definitely located in any case by placing the decimal point in the result so that there is one whole number figure in the square root for each group of two digits occurring to the left of the decimal point in the given number. The groups of two figures referred to are called periods. The last period so marked off may consist of either one or two digits. For additional description see paragraphs entitled Square Root of Numbers included in SECTION I.

The above rules for determining which section of the *A* scale is to be employed in finding the square root of a number are obviously not applicable to numbers which are decimal fractions, since, in this case there are no digits to the left of the decimal

point. To obtain the square root of any decimal fraction, first move the decimal point in the given number an *even* number of places to the right of its original position so that the value of the fraction becomes some number between 1 and 100. The resulting number may be a whole number, or a whole number accompanied by a decimal as shown below. Now find the square root of the number thus formed, and in the answer move the decimal point to the left one-half as many places as it was moved to the right in the first operation. The resulting number is the square root of the given decimal fraction.

$$\begin{aligned}\sqrt{0.0625} \\ \sqrt{006.25} = 2.5 \\ 0.0625 = .25 \text{ or } 0.25 \text{ Ans.}\end{aligned}$$

The operation of squaring a number can be accomplished by a reverse process to that described above. Thus, the square of any number is found at a point on the *A* scale directly above that number located on the *D* scale. For locating this point on the *A* scale, the cross-hair of the glass is invariably employed.

The method of squaring a number by the use of the *A* and *D* scales is in addition to the method of multiplication on the *C* and *D* scales in which case the given number is simply multiplied by itself.

Square Root of the Sum of the Difference of Two Squares

The need for calculating the square root of the sum of two squares or the square root of the difference of two squares is frequently encountered. If these operations are performed with the use of a slide rule, it is advisable to factor out the square root of one of the terms beneath the radical sign before the square root of the sum or the difference of the squares of the terms is obtained. This procedure makes it possible to deal with much smaller numbers with a resulting saving in time and effort. This is evident from an examination of the examples which follow.

$$\begin{aligned}\sqrt{90^2 + 360^2} &= 90 \sqrt{\frac{90^2}{90^2} + \frac{360^2}{90^2}} = 90\sqrt{1^2 + 4^2} \\ &= 90\sqrt{1 + 16} = 90\sqrt{17} = 90 \times 4.12 \\ &= 371 \\ \sqrt{120^2 - 40^2} &= 40 \sqrt{\frac{120^2}{40^2} - \frac{40^2}{40^2}} = 40\sqrt{3^2 - 1^2} \\ &= 40\sqrt{9 - 1} = 40\sqrt{8} = 40 \times 2.83 \\ &= 113\end{aligned}$$

The operations as shown above in detail are not necessarily written down when a slide rule is being used.

Cube Roots and Cubes of Numbers

On some slide rules a scale may be found which is made up of three identically graduated scales placed consecutively and at the top of the frame. The total length of these three scales is the same as the full length of one *C* or *D* scale and each are graduated in a similar manner, although the subdivisions are fewer in number and consequently

have a larger value. The three consecutively placed scales are referred to collectively as the *K* scale. It is usually distinctly marked by a letter *K* found at the left-hand edge of the rule and on a line with the scale it identifies. Each of the three divisions making up the *K* scale is referred to as K_1 , K_2 , and K_3 respectively, reading from left to right.

K_1	2	3	4	5	6	7	8	9		
D_1	1	2	3	4	5	6	7	8	9	10

The cube root of any number located on the *K* scale is found at a point on the *D* scale directly below that number. For locating this point on the *D* scale, the cross-hair of the glass is invariably employed. Since the *K* scale consists of three equal and identical sections, it becomes necessary to determine on which of the three the given number should be located. Of the several rules available for governing this decision, the most infallible seems to be:

Use K_1 for a number of 1 digit to the left of the decimal point.

Use K_2 for a number of 2 digits to the left of the decimal point.

Use K_3 for a number of 3 digits to the left of the decimal point.

The decimal point in the cube root of all numbers from 1 to 100 is placed so that there is one whole number in the root. Thus, the cube root of 5, 15, and 512 are 1.71 . . . , 2.46 . . . , and 8.00 respectively.

The above rules for determining which section of the *K* scale is to be employed in finding the cube root of a number are obviously not applicable to numbers which lie outside the range of numbers of 1 to 1000. To find the cube root of any number less than 1 or greater than 1000, it is necessary to first move the decimal point 3, 6, 9, or any multiple of 3 places to either the left or the right from its original position so that the value of the number to the left of the decimal point becomes some number between 1 and 1000. Now find the cube root of the number thus formed, and then move the decimal point one-third as many places as it was moved in the first place, but in the opposite direction. The resulting number is the cube root of the given number.

$$\begin{aligned}\sqrt[3]{0.027} \\ \sqrt[3]{0.027} = 0.00 \\ \sqrt[3]{0.027} = 0.03 \text{ Ans}\end{aligned}$$

The operation of cubing a number can be accomplished by a reverse process to that described above. Thus, the cube of any number is found at a point on the *K* scale directly above that number located on the *D* scale. For locating this point on the *K* scale the cross-hair of the glass is invariably employed.

The method of cubing a number by the use of the *K* and *D* scales is in addition to the method of multiplication on the *C* and *D* scales in which the given number is simply multiplied by itself two times.

Trigonometric Functions

The slide rule provides a convenient means by which the natural trigonometric ratios sine and tangent for any acute angle can be read directly from the appropriate scale. The values of the cosine function for the various angles are not given on the rule, but they may be indirectly determined in any case by finding the sine of the angle which is the complement of the given angle. One angle is said to be the complement of another angle if their sum is 90° . Thus, the slide rule may be said to provide all three of the ratios—sine, cosine, and tangent.

An inspection of the back of the stick shows that it is graduated with three scales, one being a scale of sines, indicated by the letter *S*; another a scale of tangents, marked *T*; and the third scale will be either a *B* or an *L* scale. The use of the *L* scale is described in later paragraphs entitled Logarithms.

The *S* and the *T* scales are both graduated in degrees and fractions of a degree, called minutes. The divisions representing degrees are clearly marked, but the subdivisions representing minutes are not. Since each subdivision may represent a number of minutes, it becomes necessary to determine the value of these graduations when reading the scales. It should be remembered that 1 degree is equal to 60 minutes. A position on the scale intermediate between two numbered graduations as 5° and 6° is read as $5^\circ 30'$. If there are, for example, six divisions between two adjacent degree graduations, then each subdivision represent $10'$.

On the *S* scale, the graduations extend from 0° to 90° , but on the *T* scale, the range of angles is from approximately 6° to 45° . This limitation of the angles on the *T* scale is made necessary as the tangent function varies over such a wide range, being 1 for 45° and becoming approximately 4000 as 90° is approached. Consequently, it would be impossible to represent the entire range with any degree of accuracy in the limited space available.

The sine or the tangent of an angle is most easily found on any slide rule by installing the stick in the rule so that the *A* or *B*, *S*, *T*, and *C* or *D* scales are all on the same side. This requirement may make it necessary to have the back side of the stick facing forward in some slide rules. The *A* and *B* scales are identical, therefore they may be used interchangeably. This is also true for the *C* and *D* scales.

Sine of an Angle

With the end graduations of the *S* scale exactly in line with the end graduations of the *A* or *B* scales, the glass may be moved to any position along the *S* scale, and the sine of the angle thus marked will be indicated by the cross-hair of the glass at a point directly above on either the *A* or *B* scale. The values of the sines found on either the *A* or *B* scale vary from 0.01 to 0.1 on the left half of the scale, and from 0.1 to 1.0, on the right half. Thus, it is important to realize that for angles having sines read on the left half of the scale (0° to $5^\circ 45'$) the decimal point must be followed by one zero when the position of the decimal point of the function is being determined. For angles having sines read on the right half of the scale, the sequence of digits representing the function immediately follows the decimal point.

Another method of finding the sine of an angle can be employed when using the

type of slide rule which does not require the stick to be reversed for the operation as previously described. With the *S* scale facing rearward in its normal position, the stick is drawn out to the right until the desired angle is in line with the cross-hair of the index mark of the part cut out of the back of the frame. The rule is then turned around and the sine is found on the *B* scale at a point directly below the right index of the *A* scale.

Cosine of an Angle

The cosine of an angle is not given directly on the slide rule, but the value of this function for any angle may be determined indirectly by finding the sine of the angle which is the complement of the given angle.

Tangent of an Angle

With the end graduations of the *T* scale exactly in line with the *C* or *D* scales, the glass may be moved to any position along the *T* scale, and the tangent of the angle thus marked will be indicated by the cross-hair of the glass at a point directly below on either the *C* or *D* scale. The sequence of digits representing the function immediately follows the decimal point as the lower limit of the *T* scale excludes all angles of such size as to have their tangents measured in hundredths. The tangents of angles below this lower limit are replaced by the corresponding sines without introducing any appreciable error. The tangents of angles larger than 45° can be found by either one of two methods.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\tan \theta = \frac{1}{\tan(90^\circ - \theta)}$$

Another method of finding the tangent of an angle can be employed when using the type of slide rule which does not require the stick to be reversed for the operation as previously described. With the *T* scale facing rearward in its normal position, the stick is drawn out to the right until the angle in question is in line with the cross-hair of the index mark of the part cut out of the back. The rule is then turned around and the tangent is found on the *C* scale at a point directly above the right index of the *D* scale.

Logarithms

The complete logarithm of any number consists of two parts called the characteristic and the mantissa. The characteristic is that part of the logarithm to the left of the decimal point, and its magnitude is determined by a mental calculation described in the paragraphs titled RULES FOR CHARACTERISTICS which is included in SECTION IV. The mantissa is the decimal part of the logarithm, or that part to the right of the decimal point. The value of the mantissa is the same for any given sequence of digits, regardless of the position of the decimal point in that sequence. Thus, the complete logarithms of 12.34 and 1234 are 1.091315 and 3.091315 respectively. It is obvious that the mantissas of these two logarithms are identical.

although the characteristics are of different value, being dependent upon the position of the decimal point in the given number.

The mantissa of the logarithm of any number is read on the *L* scale after properly locating the number on the *D* scale. The entire length of the *L* scale is graduated into major divisions of equal length and numbered from 1 to 10. Each of these graduations are subdivided into ten equally spaced minor divisions which in turn are each subdivided into ten smaller divisions. The length of the scale may, therefore, be said to be graduated decimal, and no difficulty should be encountered in determining the value of any subdivision, or any position on the scale.

Because the scales of the various slide rules are not similarly arranged, it is necessary to describe several possible methods of obtaining the mantissa for any given number.

If the *D* and the *L* scales are both found on the same side of the rule when the stick is installed with its front face forward, the mantissa is read on the *L* scale at a point directly above the position of the number on the *D* scale. The cross-hair of the glass when moved to the position of the number on the *D* scale accurately locates the mantissa directly above on the *L* scale.

A few slide rules are operated by the same method as described above after installing the stick in the frame with its back side forward. Such rules can be identified, as the numbered divisions of the *L* scale will increase from left to right when the stick is installed in the manner described.

If the *L* scale is one of the three scales found on the back side of the stick, and the numbered divisions decrease from left to right, the mantissa is read from the *L* scale when the left index of the *C* scale is placed directly above the value of the number on the *D* scale. A cross-hair is usually provided in the back and at one end of the frame of the rules which are operated in this manner. The *C* and *D* scales are obviously on the side of the rule opposite to the *L* scale; consequently the rule must be turned over to read the mantissa.

POSITION OF THE DECIMAL POINT DETERMINED BY CHARACTERISTICS

The decimal point in the result of all operations involving multiplication or division performed on the *C* and *D* scales can be definitely located by employing the characteristic of the logarithms of the numbers involved. The characteristic of each of the terms of the problem is first written down somewhere adjacent to the number. Then the slide rule work is performed according to the methods already described for multiplication, division, and combined operations. For each separate operation in the solution it is necessary to observe the position of the left index (1) of the *C* scale. If this index projects beyond the left index of the *D* scale, the characteristic of the term which caused it to extend should be increased by 1. When the index does not project, the characteristic of the term used remains unchanged. The characteristic used in placing the decimal point in the answer is the algebraic difference of the characteristics of the terms constituting the dividend (numerator) and the characteristics of the terms comprising the divisor (denominator).

For continued operations it is essential that the position of the left index of the *C* scale be noted as each new term is established on the rule, and any necessary addition to the characteristic be immediately made. Not only is the characteristic of the term in-

creased which caused the left index to first project, but also the characteristics of the following term or terms are also increased if they allow the index to remain extended to the left as such terms are employed.

$$\frac{45}{15} = \frac{1}{3} - \frac{1}{3} = 3 \text{ or } 3.$$

$$\frac{50}{75} = \frac{1}{3} - \frac{2}{666\dots} = 666\dots = .666\dots.$$

$$\frac{-1}{.25} \times \frac{1+1}{24} = \frac{(1)-(3)}{24} = \frac{-2}{24} = .024$$

Section VII
ALGEBRA — PROBLEMS

Positive and Negative Numbers

1. Find the sum of the following.

(a) -1	(b) -5	(c) -9	(d) +13
+2	-6	+10	+14
-3	+7	-11	-15
<u>+4</u>	<u>+8</u>	<u>+12</u>	<u>-16</u>

(e) -3	(f) 10 + (-14) =
+8	(g) .3 + (-.08) =
16	(h) -1.2 + 6 =
<u>-3</u>	

2. Find the difference when the lower number is subtracted from the upper number in each column.

(a) 5	(b) 3	(c) 5	(d) -5	(e) 5	(f) -8
<u>3</u>	<u>5</u>	<u>-3</u>	<u>3</u>	<u>10</u>	<u>-3</u>

3. Find the product of all the numbers in each column.

(a) -1	(b) 5	(c) -1	(d) 5
2	<u>-6</u>	-6	2
3	-7	3	-7
<u>4</u>	<u>8</u>	<u>-4</u>	<u>8</u>

4. Find the quotient of the following.

(a) $\frac{16}{-4}$	(b) $\frac{-12}{-3}$	(c) $\frac{144}{-16}$	(d) $\frac{56}{8}$	(e) $\frac{175}{-5}$	(f) $\frac{144}{-6}$
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Common Fractions

5. *Addition*

(a) $\frac{2}{5} + \frac{3}{5} =$	(b) $\frac{2}{3} + \frac{3}{4} =$	(c) $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} =$	(d) $\frac{x}{4} + \frac{y}{6} + \frac{z}{12} =$
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6. *Subtraction*

(a) $\frac{3}{5} - \frac{2}{5} =$	(b) $\frac{5}{6} - \frac{1}{3} =$	(c) $\frac{3}{4} - \frac{1}{8} =$	(d) $\frac{x}{5} - \frac{y}{3} =$
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7. *Multiplication*

(a) $\frac{1}{2} \times \frac{2}{3} =$	(b) $2 \times \frac{5}{4} =$	(c) $\frac{13}{25} \times \frac{25}{13} =$	(d) $\frac{x}{4} \times \frac{2}{x} =$
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8. *Division*

(a) $\frac{3}{5} \div \frac{4}{5} =$	(b) $\frac{5}{4} \div 2 =$	(c) $\frac{6}{5} \div 2 =$	(d) $\frac{x}{2} \div \frac{x}{4} =$
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9. Find the reciprocals of the following.

$$(a) \frac{1}{\frac{2}{3}} = \quad (b) \frac{1}{\frac{1}{3}} = \quad (c) \frac{1}{\frac{1}{3} + \frac{1}{2}} = \quad (d) \frac{1}{\frac{a}{b}} =$$

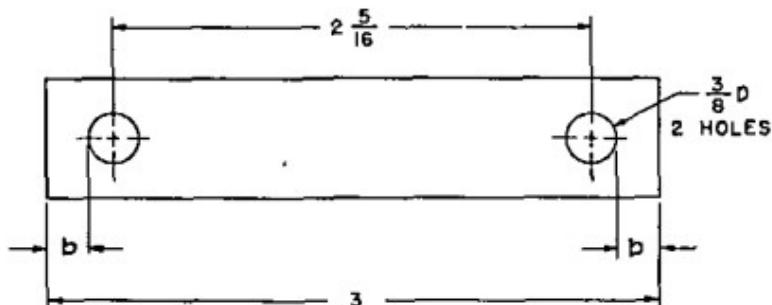
10. Least common denominator.

$$(a) \frac{5}{24} + \frac{3}{15} + \frac{2}{9} = \quad (b) \frac{2}{3} + \frac{1}{4} - \frac{1}{6} = \quad (c) \frac{x}{a} + \frac{y}{b} = \quad (d) \frac{x}{a} - \frac{y}{b} =$$

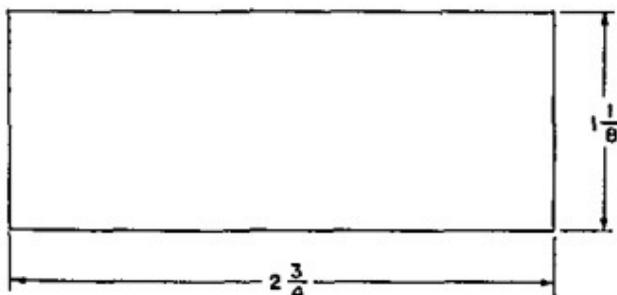
11. Mixed numbers.

$$(a) 1\frac{1}{2} + 2\frac{3}{4} = \quad (b) 5\frac{1}{8} - 3\frac{3}{4} = \quad (c) 2\frac{2}{3} \times 1\frac{1}{8} = \quad (d) 12\frac{1}{3} \div 3\frac{1}{2} =$$

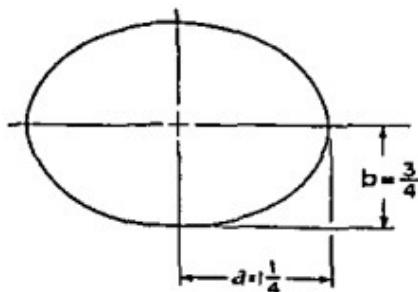
12. Find the dimension (b).



13. Find the area of the rectangle.



14. Find the area of the ellipse with a semi-major axis (a) of 1 1/4" and a semi-minor axis (b) of 3/4".



15. The gross (total) weight of a given airplane is 10,000 lbs., its weight empty is 60% of its gross weight. The crew weighs 170 lbs., fuel and oil weigh 1,000 lbs. and remainder is to be freight. How much freight can be carried by the airplane.

Decimal Fractions

16. Find the *sum* of all the numbers in each column.

(a) .125	(b) .09375	(c) 2.25	(d) 0.125
.250	.0625	.032	2.500
<u>.375</u>	<u>.345</u>	<u>3.278</u>	<u>1.075</u>

17. Find the difference when the lower number is subtracted from the upper in each column.

(a) 2.50	(b) 3.75	(c) 1.118	(d) 1.50	(e) .25	(f) 6.25
<u>.125</u>	<u>1.0625</u>	<u>.219</u>	<u>.15</u>	<u>.125</u>	<u>3.875</u>

18. Find the product of all the numbers in each column.

(a) 5.625	(b) 3.1416	(c) 3.25	(d) 17.25
<u>.05</u>	<u>.25</u>	<u>1.25</u>	<u>3.02</u>

19. Find the quotient when the upper number is divided by the lower.

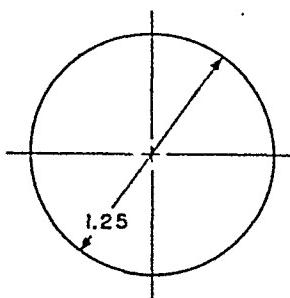
(a) <u>3.1416</u>	(b) <u>2.5</u>	(c) <u>22.7766</u>	(d) <u>23.625</u>
<u>4</u>	<u>.0625</u>	<u>3.1416</u>	<u>1.05</u>

20. Find the decimal equivalents of the following.

(a) $\frac{1}{3} =$	(f) $\frac{1}{8} =$
(b) $\frac{1}{4} =$	(g) $\frac{1}{9} =$
(c) $\frac{1}{5} =$	(h) $\frac{1}{16} =$
(d) $\frac{1}{6} =$	(i) $\frac{1}{32} =$
(e) $\frac{1}{7} =$	(j) $\frac{1}{64} =$

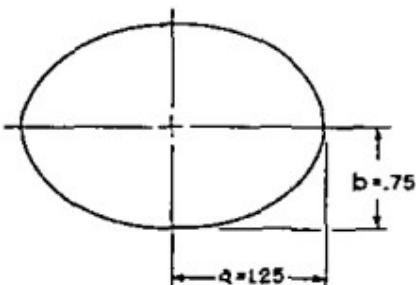
21. Find the circumference of a 1.25 inch diameter circular shaft.

$$\begin{aligned}\text{Circumference} &= \pi D = 3.1416 \times 1.25 \\ \pi &= 3.1416 \text{ (approx.)}\end{aligned}$$



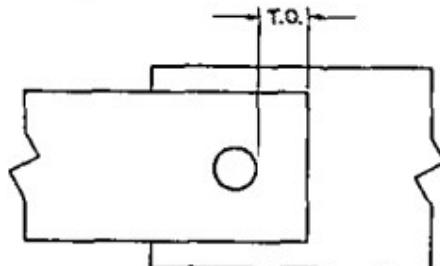
SOLUTION OF EQUATIONS

22. Find the area of an ellipse with a semi-major axis (a) of 1.25", and a semi-minor axis (b) of .75"



$$\text{Area} = \pi ab = 3.1416 \times 1.25 \times .75$$

23. The tear-out strength (P_t) of a riveted or bolted joint can be computed by the use of the formula



$$P_t = F_s \times 2T.O. \times t$$

Where P_t = tear-out strength of sheet (lbs.)

F_s = allowable shear stress of sheet material (lbs./sq. in.)

$T.O.$ = tear-out distance or the distance from the edge of the rivet or bolt hole to edge of the sheet (ins.)

t = thickness of sheet (ins.)

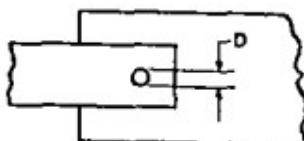
Find the tear-out strength of a lap joint for which

$$F_s = 34,000$$

$$t = .040$$

$$T.O. = 25$$

24. The bearing strength (P_{br}) of a riveted or bolted joint can be computed by the formula



$$P_{br} = F_{br} \times D \times t \quad \text{Where } P_{br} = \text{bearing strength of sheet (lbs.)}$$

F_{br} = allowable bearing stress of sheet material (lbs./sq. in.)

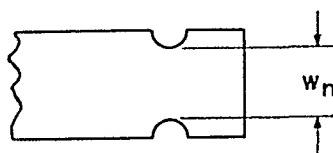
Find the bearing strength of a joint for which

$$F_{br} = 75,000 \text{ p.s.i.}$$

$$D = 3/32$$

$$t = .040$$

25. The tensile strength (P_t) of a riveted or bolted joint can be computed by the use of the formula



$$P_t = F_t \times W_n \times t$$

Where P_t = tensile strength of sheets (lbs.)

F_t = allowable tensile stress of sheet material (lbs./sq. in.)

W_n = net width of sheet (in.)

t = thickness of sheet (ins.)

Find the tensile strength of a joint where

$$F_t = 56,000 \text{ p.s.i.}$$

$$t = \frac{1}{2}''$$

$$W_n = .10$$

Square Root of Numbers

26. Solve the following.

$$(a) \sqrt{17424}$$

$$(d) \sqrt[4]{256}$$

$$(b) \sqrt{11296321}$$

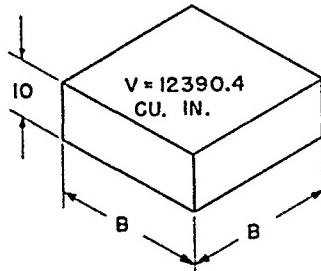
$$(e) \sqrt{\frac{36}{64}}$$

$$(c) \sqrt[3]{8}$$

27. Find the radius of a circle which has an area of 28.2744 sq. in.

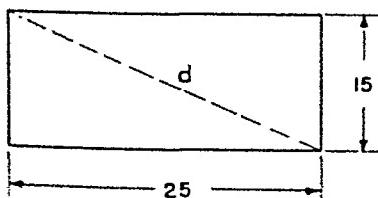
$$\text{Area} = \pi r^2 = 3.1416 R^2$$

28. Find the length of the side of a box which has a square bottom and a height of 10 inches. The volume of the box is 12390.4 cu. in.



$$\text{Volume} = B \times B \times 10$$

29. The length of the diagonal of a rectangle is given by the Pythagorean Theorem as $d = \sqrt{a^2 + b^2}$ where a and b are the dimensions of the two sides. Find the length of the diagonal when $a = 15$ inches and $b = 25$ inches.



$$d^2 = 15^2 + 25^2$$

$$d = \sqrt{15^2 + 25^2}$$

SOLUTION OF EQUATIONS

30. Find the length of the side of a square which has a diagonal of 14.14 inches.

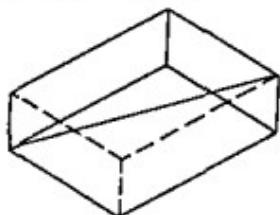


$$14.14^2 = L^2 + L^2 = 2L^2$$

$$L^2 = \frac{14.14^2}{2}$$

$$L = \sqrt{\frac{14.14^2}{2}}$$

31. The length of the longest diagonal of a rectangular solid is given by the formula $d = \sqrt{a^2 + b^2 + c^2}$ where a , b , and c are the dimensions of the three sides of the solid. Find the length of the longest diagonal when $a = 8$ inches, $b = 12$ in., $c = 16$ in.



$$d = \sqrt{a^2 + b^2 + c^2}$$

Exponents

32. Find the product of all factors in each line and the numerical product.

(a) $(x)(x) =$ Let $x = 3$

(b) $(x^1)(x^2) =$ Let $x = 2$

(c) $(x^4)(x^3) =$ Let $x = 2$

(d) $(x^5)(x^6) =$ Let $x = 3$

33. Solve the following.

(a) $\frac{3x^2}{2} + \frac{8x^2}{5} =$ (c) $(5y)^3 =$

(b) $(2x)^2 =$ (d) $\frac{(6y)^2}{-3y} =$

34. Find the product of all factors in the following.

(a) $(x)(x^1)(x^2)(x^3) =$ (d) $(2x)(3x^2)(4x^3)(5x^4)$

(b) $(x^2)(x^3)(x^4)(x^5) =$ (e) $\frac{(x)}{2} \frac{(3x^2)}{4} \frac{(5x^3)}{2} \frac{(4x^4)}{5}$

(c) $(x^5)(x^{12})(x^6)(x^4) =$

35. Simplify the following.

(a) $\frac{x^5}{x^2} =$ (c) $\sqrt{16x^4y^2}$

(b) $\frac{x}{x^{-2}} =$ (d) $\sqrt[3]{\frac{8x^4y^5}{x^{-5}y^2}} =$

36. Solve the following.

(a) $\sqrt{a^2} =$ (c) $(4a^2)b^2 =$

(b) $\sqrt[3]{y^7} =$

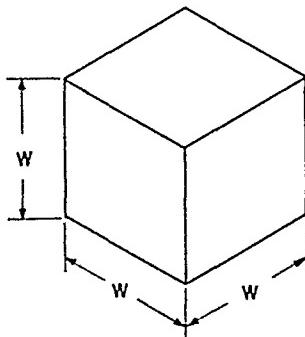
37. (a) $(x+y)(x+y) =$ (d) $(x)(x)^6 =$
 (b) $\sqrt{8x^2 + (3x)(-2x)} =$ (e) $(2^3)(3)^2 =$
 (c) $(a^2)(a^3) =$ (f) $3^2 - 4^3 =$

38. (a) $x^3 \div x^2$ (d) $a^{2.5} \div a^{.5} =$
 (b) $4^3 \div 2^5$ (e) $(x^{1.5})(x^2) =$
 (c) $\frac{\sqrt{x^3}}{\sqrt{x^5}} =$

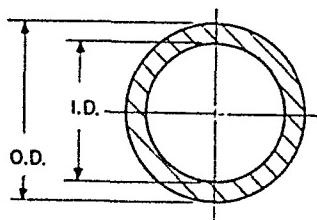
39. (a) $\sqrt[4]{8^4} =$ (c) $\sqrt[4]{2^8} =$
 (b) $\sqrt[3]{7^4} =$ (d) $\sqrt[5]{a^n} =$

40. (a) $(xy)^2 (xy^4) =$ (d) $-4^2 \div 2^4 =$ (g) $\frac{a^2}{a^{-3}}$
 (b) $(3a)^2 (3^2a) =$ (e) $a^2 \div b^2 =$
 (c) $a^3b^3 \div a^5b^5 =$ (f) $\frac{x^{-3}}{y^{-2}} =$

41. (a) Find the surface area of a cube which measures (w) on any side.
 (b) If $w = 6$, find the surface area of the cube.



42. Find the cross sectional area of a one inch circle having a wall thickness of .0625 in.



$$\text{Area of Circle} = \pi R^2 = \pi \left(\frac{D}{2}\right)^2 = .7854 D^2$$

$$\text{Since } R = \frac{D}{2}, R^2 = \left(\frac{D}{2}\right)^2 = \frac{D^2}{4}$$

$$\text{Area of tube} = \frac{\pi \overline{O.D.^2}}{4} - \frac{\overline{I.D.^2}}{4} = .7854 \overline{O.D.^2} - .7854 \overline{I.D.^2}$$

SOLUTION OF EQUATIONS

Solution of Equations

Solve for the unknown (x) in each of the following equations. Check the results obtained to see that the given equation is satisfied.

43. $x - 2 = 0$

48. $x - 6 = 4x$

44. $x + 3 = 18$

49. $x + 2 = 2x - 6$

45. $x - 5 = 3$

50. $x^2 + 5x - x^2 - 4x - 4 = 4$

46. $2x + 7 = 14 + x$

51. $x^2 - 3 + 2x = x^2 - x$

47. $5x = 10 + 4x$

52. $x^2 + 2x - 5 = x^2 + x$

Multiplication—Division

53. $2x = 10$

57. $4x + 3 = 6x - 8$

54. $5 = \frac{x}{2}$

58. $3x + 3 = 18$

55. $3 = \frac{12}{x}$

59. $6x - 4 = 4x + 6$

56. $\frac{2x}{3} = 6$

60. $5x = 35 - 2x$

61. $5x = 0$

62. $4x^2 = 44x$

63. The aspect ratio of a monoplane airplane wing is expressed by the formula:

$$\overline{AR} = \frac{b^2}{s}$$

Where \overline{AR} = The aspect ratio (a non-dimensional number)

b = Wing span in feet

s = Wing area in square feet

For an aspect ratio of 10, find the required span if the wing area equals 300 square feet

Method of Solution—Single Equation

Simplification—solve for x in each of the following

64. $33x = (30)(60) - (5)(48) - (10)(24)$

65. $10x - (x - 9) = 35 - 4(2x - 1)$

65. $x - (3 - 2x) = 2x - 4$

67. (a) $3(x + 1) + 4 = 5(x - 3)$

(b) $6(x - b) = 12 + 5(x + b)$

68. (a) $12x = \frac{x - 6}{3}$

(b) $\frac{12}{x - 3} = \frac{4}{x + 1}$

(c) $(2x - 1) - 2 [4v - (x + 2) + 10] = 8x + 1$

69. (a) $x + (3 - a) = 3x + (1 + 2a)$

(b) $a - (x + 2) - (a + 3) = x + 4(a + 1) - (x + 5)$

70. (a) $\frac{2}{3}x + \frac{3}{4}x = 4$ (b) $3 + \frac{2x + 4}{4x - 2} + \frac{2}{x + 2} = 6$

71. The equation for finding the bending stress f_b in a beam is

$$f_b = \frac{MC}{I}$$

In a beam of rectangular cross section $I = \frac{bh^3}{12}$ and $c = \frac{b}{2}$

Write the complete equation for computing the stress in this beam in terms of b and h .

72. The aspect ratio (AR) of the wing of airplane $= \frac{b^2}{s}$ where $s = bc$.

Write the equation for AR .

73. The formula for converting the readings of Fehrenheit temperature to Centigrade is:

$$C^\circ = \frac{5}{9}(F^\circ - 32^\circ)$$

Find the temperature in Centigrade degrees which is equivalent to $86^\circ F$.

74. The formula for converting the readings of Centigrade temperatures to Fehrenheit temperatures is:

$$F^\circ = \frac{9}{5}C^\circ + 32^\circ$$

Find the temperature in Fehrenheit degrees which is equivalent to $40^\circ C$.

75. The net cross sectional area of a circular tube can be computed by the use of the following formula:

$$A = \pi t(O.D. - t)$$

Where A = cross sectional area

$\pi = 3.1416$ (approx.)

$O.D.$ = outside diameter of tube

t = wall thickness of tube

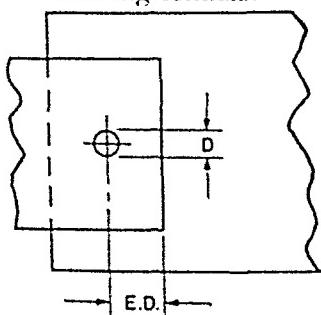
Note: Median Diameter $= O.D. - t$

Circumference of Median Line $= \pi(O.D. - t)$

Area $= \pi(O.D. - t)t = \pi t(O.D. - t)$

Find the net cross sectional area of a .75 $O.D.$ \times .065 tube.

76. The tear-out strength of a riveted joint can be computed by the use of the following formula:



$$P_s = 2F_s t \left(E.D. - \frac{D}{2} \right)$$

Where P_s = Tear-out strength of sheet

F_s = Allowable shear stress of sheet

t = Thickness of sheet

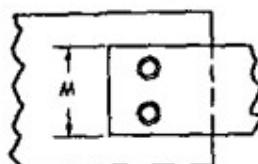
$E.D.$ = Distance from center of rivet to edge of sheet.

D = Diameter of rivet

Find the tear-out strength of a riveted lap joint for which

$$F_s = 34,000 \quad t = .040 \quad E.D. = 25 \quad D = \frac{3}{16}$$

- 77 The tensile strength of a riveted joint can be computed by the use of the following equation:



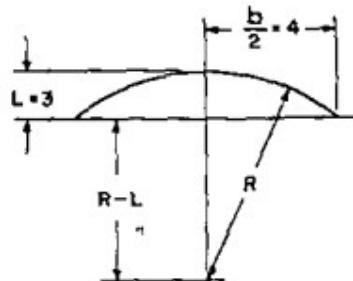
$$P_t = F_t t (W - nD)$$

Where
 P_t = net tensile strength of sheet
 F_t = allowable tensile stress in sheet
 t = thickness of sheet
 W = width of sheet
 n = number of rivets
 D = diameter of rivets

Find the net tensile strength of a riveted lap joint for which:

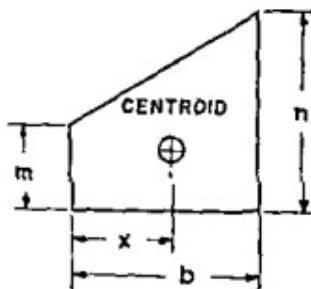
$$F_t = 56,000 \quad t = .040 \quad W = 1.5 \quad n = 2 \quad D = \frac{3}{16}$$

- 78 Find R if $L = 3$ and $\frac{b}{2} = 4$ Hint. $(R - L)^2 + \left(\frac{b}{2}\right)^2 = R^2$



79. The centroid (center of gravity) of a trapezoid may be found by the following

$$x = \frac{b}{3} \frac{m + 2n}{m + n}$$



Find x if $m = 2$ $n = 4$ $b = 5$

Trial and Error

80. $x^4 - 5x^2 + 4 = 0$

81. $x^2 - 3x + 2 = 0$

82. $\frac{x+1}{2} + \frac{x+3}{4} = 5$

83. $\frac{8x+5}{14} + \frac{7x-3}{6x+2} = \frac{4x+6}{7}$

84. $32 = \frac{64}{\sqrt{x+1}}$

Graphical Solution

85. $x^2 + x - \frac{11}{4} = 0$
86. $x^2 + 4x + 4 = 0$ (Multiple Roots)
87. $x^3 - 6x = 0$
88. $4x^3 - 40x = 0$
89. $3x^4 - 14x + 8 = 0$ (Two roots are imaginary)
90. $r^3 - r^2 + r - 2 = 0$ (Two roots are imaginary)
91. $2x^3 - 15x + 10 = 0$
92. $x^3 - 4x^2 - 5x + 14 = 0$
93. $x^3 - 2x^2 - 7x - 4 = 0$
94. $x^4 - 4x^3 - 4x^2 + 16x + 15 = 0$ (Multiple roots)

Factoring

Factor and solve the following:

95. $x^2 + 2x + 1 = 0$
96. $x^2 - 3x - 4 = 0$
97. $x^2 + x - 3 = 0$
98. $4x^2 + 4x = -1$
99. $3y^2 - 5y + 2 = 0$
100. $5t^2 + 19t + 12 = 0$
101. $5t^2 + 12 = 19t$
102. $2m^3 + 5m^2 - 4m - 3 = 0$

Expand the following:

103. $(x - 1)(2x + 3)$
104. $(2n + 3)(2n - 1)$
105. $(y + 5)(5y + 3)$
106. $(x + 3)(2x + 6)(4x - 5)$
107. $(3x + 4)^2$
108. $(x + 5)^3$
109. $(\sqrt{3x+9})(\sqrt{3x+9})$
110. $(x - \sqrt{2x+3})^2$
111. $(x + \sqrt{x^2 + 4x + 3})^2$
112. $\frac{2(3x^3 - x^2 - 12x + 4)}{2x + 4}$
113. $\frac{10x^4 + 19x^3 - 6x^2 - 12x + 9}{2x + 3}$
114. $\frac{7x^3 - 4x^2 + 6x - 9}{x^2 - 3x + 2}$
115. $\frac{3x^3 + 4x^2 - 6x - 8}{x^2 - 2}$
116. $\left(a^2 - \frac{2a}{3} + \frac{1}{4}\right) \left(2a^2 - \frac{a}{3} - \frac{1}{2}\right)$
117. $\left(\frac{b^2 - b - 1}{3}\right) \left(\frac{b^2 - 2b - 5}{6}\right)$

118. The product obtained by multiplying the sum of two integers by the difference between the same integers is 27. If the larger number is twice as large as the smaller number, what are the two numbers? Let x represent the smaller number.
119. The area between the circumference of two circles, one within the other, is given by the formula:

$$A = \pi (R_1 + R_2)(R_1 - R_2)$$

Where R_1 = Radius of outer circle
 R_2 = Radius of inner circle

The circles need not be concentric, that is, drawn from the same center. Find the area between the circumference of two circles having diameters of 4 inches and 3 inches.

SOLUTION OF EQUATIONS

Completing the Square

120. $x^2 + 10x - 39 = 0$

125. $3x^2 + 121 = 44x$

121. $x^2 - 5x + 6 = 0$

126. $4x^2 - 4x = 7$

122. $4x^2 + 4x - 54 = 45$

127. $3x^2 + 9x - 4 = 0$

123. $6x^2 + 5x = 1$

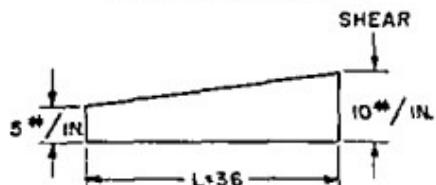
128. $9x^2 + 6x - 17 = 0$

124. $7x - 6 - 2x^2 = 0$

129. $6x^2 - x = 12$

Quadratic Equations

130. The bending moment is a maximum at the location along the span of a beam where the shear is zero. The point, x , of zero shear for a simple beam load as shown is.

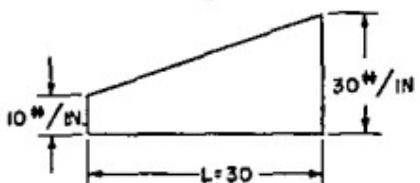


$$120 = 5x + \frac{10 - 5}{36} \frac{x^2}{2} \quad \text{or}$$

$$x^2 + 72x - 1728 = 0$$

Find the point of zero shear.

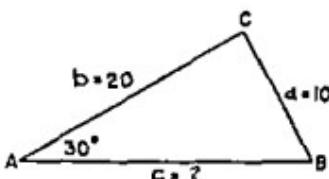
131. Find the point x of zero shear from the equation:



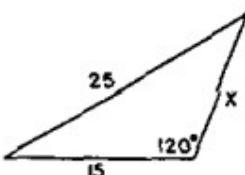
$$250 = 10x + \frac{30 - 10}{30} \frac{x^2}{2} \quad \text{or,}$$

$$x^2 + 30x - 750 = 0$$

132. $a^2 = b^2 + c^2 - 2bc \cos A$
 $10^2 = 20^2 + c^2 - (2 \times 20 \times c \times .866)$
 Find the true length of side c .



133. $25^2 = 15^2 + x^2 - (2)(15)(x)(-\cos 120^\circ)$
 Find the length of side x .



134. The allowable column stress for a steel tube is given by the formula.

$$F_{cr} = 135,000 - 15.92 \frac{(L/I)^2}{\rho} \quad \text{Short Column}$$

$$F_{cr} = \frac{286 \times 10^6}{(L/\rho)^2} \quad \text{Long Column}$$

The critical or transition $\frac{L'}{\rho}$ is that slenderness ratio above which columns are long and below which columns are short. At this slenderness

ratio, the allowable stress given by either of the above formulas is the same. A graph of the two equations is tangent at this point; find the critical slenderness ratio L'/ρ for tubes made of alloy steel designed by the above formulas. Also determine the allowable column stress, F_{co} at the critical L'/ρ .

135. The tear-out strength (P_s) of a riveted joint is given by the formula:

$$P_s = 2 \times T.O. \times t \times F_s$$

and the bearing strength (P_{br}) is given by:

$$P_{br} = D \times t \times F_{br}$$

Where $T.O.$ = distance from edge of hole to edge of sheet

F_s = allowable shear stress of sheet

t = thickness of sheet

D = diameter of rivet

F_{br} = allowable bearing stress of sheet

Find the minimum tear-out distance for a riveted joint so that the tear-out strength and bearing strength are equal.

Assume $a = .064$

$$F_s = 34,000$$

$$D = \frac{3}{16}$$

$$F_{br} = 82,000$$

Quadratic Equations (Formula)

136. $x^2 + 4x + 1 = 0$

137. $5x^2 + 10x + 15 = 30$

138. $x^2 + 4x = 2x + 3x^2 - 8$

139. $65x^2 + 96x + 36 = 256x^2$

140. $x^2 + \left(\frac{5-x}{3}\right)^2 = 4$

141. $\frac{24}{x+9} + \frac{24}{x-9} = 6$

142. $x^2 - 0.3x + 1.8 = 0$

143. $x^2 + .25x = 17$

144. $x^2 - 1.25x - 1 = 0$

145. $3x^2 - 0.1x - 0.33 = 0$

Radical Equations

146. $x + \sqrt{x-5} = 11$

147. $2\sqrt{x+1} + x = 7$

147. $\sqrt{2x+29} - \sqrt{x+6} = 3$

149. $\sqrt{x-1} = 3 + \sqrt{x-10}$

150. $x+5 - 2\sqrt{x+5} + 1 = 3x+4$

151. $3x+5 = 2 + \sqrt{3x+4}$

152. $\sqrt{3x-2} - \sqrt{x+3} = 1$

153. $\sqrt{2x+5} - \sqrt{x-6} = 3$

154. $\sqrt{x+6} - 1 = \sqrt{3x+7}$

155. $\sqrt{x+16} - 3 = \sqrt{x-5}$

Simultaneous Equations

Graphical.

156. $\begin{array}{l} x+y=4 \\ x-y=2 \end{array}$

157. $\begin{array}{l} x+2y=3 \\ x-2y=2 \end{array}$

158. $\begin{array}{l} 2y-x=6 \\ 2x+y=8 \end{array}$

159. $\begin{array}{l} x-5y+4=0 \\ 3x-10y+2=0 \end{array}$

160. $\frac{x}{3} + y = 10$

$$3 + \frac{x}{3} = y - 3$$

161. $2x + y = 1$

$$4x + y^2 = 17$$

162. $4x^2 - y^2 = 15$

$$2x - y = 3$$

163. $\frac{2x + y}{x^2 + y^2} = 1$

$$\underline{x^2 + y^2 = 1}$$

164. $9x^2 - 4y^2 = 35$

$$\underline{3x + 2y = 7}$$

165. $5x^2 + 10y = 85$

$$\underline{2x^2 - 2y = 10}$$

Substitution

166. $5x + 2y = 4$

$$\underline{7x - 3y = 23}$$

167. $4x^2 - y^2 = 15$

$$\underline{2x - y = 3}$$

168. $x + y = 4000$

$$\underline{.06x + .05y = 230}$$

169. $.5A + 866B = 100$

$$\underline{866A - 58B = 0}$$

170. $y^2 - x^2 = 20$

$$\underline{x = \frac{2}{3}y}$$

171. $x^2 + xy + y^2 = 30$

$$\underline{x + y = 2}$$

172. $x^2 + y^2 = 25$

$$\underline{x + 2y^2 = 34}$$

173. $\frac{1}{x} + \frac{1}{y} = 5$

$$\underline{\frac{1}{x} - \frac{1}{y} = 3}$$

174. $2x + 3y - 4z = -1$

$$\underline{x - 6y + 2z = 3}$$

$$\underline{4x - 3y + 8z = 5}$$

175. $2x - 4y - 6z = -6$

$$\underline{3x + 5y + 4z = 16}$$

$$\underline{-x + 2y + z = 0}$$

Addition or Subtraction.

176. $5x + 4y = 22$

$$\underline{3x + y = 9}$$

177. $R_1 + R_2 - 50 = 0$

$$\underline{10R_1 - 10R_2 + 250 = 0}$$

178. $x^2 + y^2 = 5$

$$\underline{x^2 - y^2 = 3}$$

179. $\frac{10}{x} - \frac{9}{y} = 2$

$$\underline{\frac{8}{x} - \frac{15}{y} = -1}$$

180. $\frac{3x}{2} - \frac{4y}{3} = -1$

$$\underline{\frac{2x}{3} - \frac{7}{4} = \frac{7}{12}}$$

181. $BX + Y = B$

$$\underline{AX + 2Y = 2A}$$

182. $.866T - .5c = 100$

$$\underline{1.732T + 3.0c = 0}$$

183. $x + y = 23$

$$\underline{y + z = 25}$$

$$\underline{z + x = 24}$$

184. $2x + 3y = 4z$

$$\underline{3x - 4y = 5z + 4}$$

$$\underline{5x - 3z = y - 2}$$

185. $2x - y + z = 5$

$$\underline{3x + 2y + 3z = 7}$$

$$\underline{4x - 3y - 5z = -3}$$

Comparison.

186.
$$\begin{array}{r} 2x - 4y = 20 \\ 4x - 2y = 16 \end{array}$$

191.
$$\begin{array}{r} y = .577(100 - x) \\ y = 1.1917x \end{array}$$

187.
$$\begin{array}{r} 2x + 4 = 6 \\ x^2 - y = 1 \end{array}$$

192.
$$\begin{array}{r} 81 = (12 - x)^2 + y^2 \\ 36 = x^2 + y^2 \end{array}$$

188.
$$\begin{array}{r} s - 44T = 0 \\ s = 4T^2 \end{array}$$

193.
$$\begin{array}{r} x^2 - 2y - 1 = y \\ x - y + 3 = 0 \end{array}$$

189.
$$\begin{array}{r} x^2 - y^2 = 25 \\ 3x = 16y \end{array}$$

194.
$$\begin{array}{r} y = \frac{8}{(x - 2)(x - 1)} \\ y(x - 3)(x - 1) = 7 \end{array}$$

190.
$$\begin{array}{r} 5x + y = 6 \\ 2(x + 5)^2 = y^2 - 34 \end{array}$$

195.
$$\begin{array}{r} .25A - .2B = 6 \\ 3A + 4B = 8 \end{array}$$

Three Variables.

196.
$$\begin{array}{r} 2x + y - z = 1 \\ x + y + z = 6 \\ x + 2y - z = 0 \end{array}$$

199.
$$\begin{array}{r} 2x + 3y - 4z = -1 \\ x - 6y + 2z = 3 \\ 4x - 3y + 8z = 5 \end{array}$$

197.
$$\begin{array}{r} 4x - 3y + 2z = 1.5 \\ x - 6y + 4z = -.5 \\ 3x - 2y - z = \frac{7}{12} \end{array}$$

200.
$$\begin{array}{r} 2x + 7y = 48 \\ 5y - 2x = 24 \\ x + y + z = 10 \end{array}$$

198.
$$\begin{array}{r} 2x - 3y - z = -4 \\ 3x + y + 2z = 7 \\ 4x - 2y + 2z = -1 \end{array}$$

Division.

201.
$$\begin{array}{r} x^2 + xy = 20 \\ x + y = 10 \end{array}$$

203.
$$\begin{array}{r} S = 4T^2 \\ S - 44T = 0 \end{array}$$

202.
$$\begin{array}{r} A^2 + AB = 45 \\ A + B = 9 \end{array}$$

204.
$$\begin{array}{r} y = x^2 - 2x - 1 \\ y = x + 3 \end{array}$$

205.
$$\begin{array}{r} L \sin \theta = \frac{WV^2}{GR} \\ L \cos \theta = W \end{array}$$

Tan θ = ? Note Tan θ = $\frac{\sin \theta}{\cos \theta}$

206. The sum of two angles of a triangle is 120° . Find the angles if $2/3$ of one angle plus $4/5$ of the other angle is 90° .

207. The sum of two acute angles is 90° . Find each of the two angles if the larger angle exceeds the smaller by 120° .

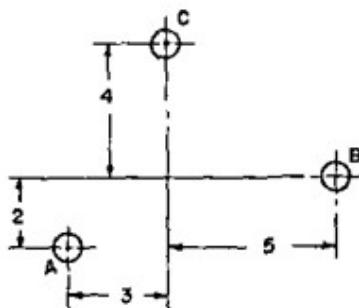
208. The sum of the three interior angles of any triangle is 180° . Find the three angles if the sum of A and B is 90° more than C , and the sum of A and C is 70° more than $2B$.

209. In locating and boring holes in a certain drill jig, it is necessary to know the diameters of three circular holes which are tangent two and two, that is, x is tangent to y and z . The distances between centers are:

$$\begin{aligned}x \text{ to } y &= 3 \\x \text{ to } z &= 4 \\y \text{ to } z &= 5\end{aligned}$$

Find the diameter of each hole.

210. The location of hole centers is usually shown by the use of rectangular coordinates as in the accompanying diagram. Find the diameter of holes to be drilled at these points if the circles are to be tangent two and two, that is, A is tangent to B and C .

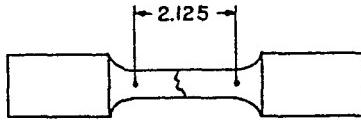
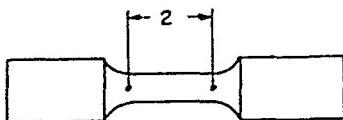


Ratio, Proportion and Variation

Percentage

211. The chemical composition of 24S aluminum alloy is 4.2% copper, 1.5% manganese, 0.5% magnesium, 93.8% aluminum. The density of the material is 173 pounds per cubic foot. Find the number of pounds of each element in one cubic foot of this aluminum alloy.
212. The chemical composition of 17S aluminum is 4% copper, 0.5% manganese, 0.5% magnesium and 95% aluminum. The density of the material is 174 pounds per cubic foot. Find the number of pounds of each element in one cubic foot of this aluminum alloy.
213. The yield point strength of aluminum alloy stiffeners can be increased by prestretching. Find the amount of prestretching (inches) required to increase the length of a stringer 3 1/2% if the original length of the stringer is 80 inches.
214. The yield point stress of 24ST extruded shapes in tension is 38,000 pounds per square inch. The ultimate tensile stress of the same material is 58,000 pounds per square inch. Find the percentage of the ultimate tensile strength which is usable without any yielding of the material.
215. The pay load of an airplane is that part of the useful load from which revenue is derived. Find the percent of a gross weight of 48,000 pounds which is represented by a pay load of 15,000 pounds.

216. The percentage elongation of a material is the difference in the gauge length before the test specimen is subjected to any stress and after rupture, expressed in percentage of the original gauge length.



Find the percentage elongation of a material which has a distance between gauge points of 2.125 after rupture if the original gauge length was 2 inches.

217. The percentage reduction of area of a material is the difference between the original cross-sectional area and the least cross-sectional area after rupture, expressed as a percentage of the original cross-sectional area. Find the percentage reduction of area of a material which has a minimum diameter of the test specimen of .450 after rupture, if the original diameter of the test specimen was .505.

Ratio.

218. The specific gravity of a substance is the ratio of the weight of the substance compared to the weight of an equal volume of pure water. The density of pure water is 62.4 pounds per cubic foot. Find the specific gravity of the following commonly used aircraft materials:

MATERIALS	DENSITY OF MATERIAL	DENSITY OF WATER	SPECIFIC GRAVITY
SPRUCE	27	62.4	
MAGNESIUM	109		
24 ST	173		
17 ST	174		
STEEL	490		

219. Find the ratio of the weight of steel to the weight of aluminum alloy 24ST. Use the densities tabulated in the above example.
220. The wing loading of an airplane is the ratio of the gross weight of the airplane to the wing area. Find the wing loading, if the gross weight is 82,500 pounds and the wing area is 3,000 square feet.
221. The power loading of an airplane is the ratio of the gross weight to the engine horsepower. Find the power loading if the gross weight is 56,000 pounds and the engine horsepower is 4,800.
222. The shear strength of an AN4 steel aircraft bolt ($\frac{1}{4}$ dia.) is 3,680 pounds and the shear strength of an AN8 steel aircraft bolt ($\frac{1}{2}$ dia.) is 14,720 pounds. Find the ratio of the shear strengths of these two bolts. Also compare the ratio of the diameters of the bolts with the ratio of the areas of the bolt shanks.

SOLUTION OF EQUATIONS

223. Given: 254 centimeters \approx 1 in.226. Given: Pressure (inches of mercury) \approx .49116
(pounds per sq. in.)

Inches	Cm
30	
25.4	
	58
	20

224 Given Area $= .7854 D^2$

Inches Hg	Pounds (sq. in.)
	14.7
42	
	100
25	

227 Given: 2π Radians \approx 360°
 $\pi = 3.1416$

Dia (Ins)	Area (sq in)
5	
	16
5	
	40

225 Given. 60 MPH Velocity $=$
88'/Sec Velocity

Degrees	Radians
30	
57.3	
	.75
	1.2

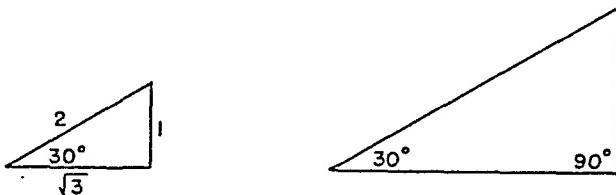
M.P.H	Ft/sec.
100	
	232
40	
	180

Miles	Kilometers
10	
	25
576	
	3050

Inverse Proportion

- 229 The rotation velocities of two gears which are meshed together are inversely proportional to the number of teeth on each gear. Find the R.P.M. of a 32-tooth gear that is driven by a 24-tooth gear which is turning at 600 R.P.M.
230. A gear having 24 teeth drives a gear with 30 teeth. Find the required rotational speed for the 24-tooth gear if the 30-tooth gear is to turn at 1000 R.P.M.
231. The volume of a perfect gas is inversely proportional to the absolute pressure, provided that the temperature remains constant. Find the final volume (V_2) if 50 cubic feet of gas at 15 pounds per square inch pressure is compressed to a pressure of 225 pounds per square inch. The temperature is assumed to remain constant.
232. The time required to travel a specified distance varies inversely as the velocity of motion. Find the rate at which an airplane must fly to travel a certain distance in 2 hours and 20 minutes, which can be flown in 3 hours and 30 minutes at a velocity of 120 M.P.H.

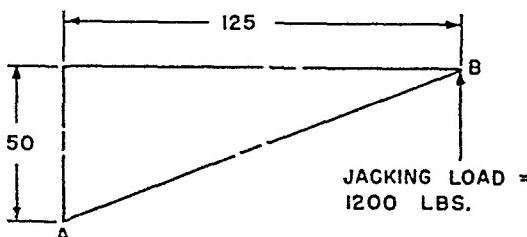
233. The landing speed of an airplane is inversely proportional to the square root of the density of the atmosphere. Find the landing speed of an airplane at an altitude of 10,000 feet which normally lands at 60 M.P.H. at sea level. The density of air decreases at an approximate rate of 2% per 1000 feet.
234. The shear strength of a rivet is proportional to the cross-sectional area of the shank. The shear strength of a $3/16$ A17S rivet is 745 pounds. Find the shear strength of a $1/8$ A17S rivet.
235. The corresponding sides of similar triangles are proportional. Find the increase in altitude in feet when an airplane climbs for 10 seconds at an air speed of 120 M.P.H. The angle of climb is 30° .



236. Interpolation, or finding a value in between two adjacent values given in a table is frequently required. Find the decimal equivalent of $9/64$ by the use of data given in the accompanying table.

$\frac{1}{8}$ OR $\frac{8}{64}$	0.125
$\frac{9}{64}$	
$\frac{5}{32}$ OR $\frac{10}{64}$.15625

237. The axial load in the lift truss of the monoplane wing can be found by proportion by an application of the following principle: The component of the load in a member, taken parallel to any axis, is to the load in the member as the length of the member projected on that axis is to the actual length of the member.



Find the axial load in the lift strut, (AB) as a result of a vertical jacking load of 1200 pounds applied at the strut attachment point.

Variation

- 238 If $F = Ma$ Where

F = Force

M = Mass or $\frac{w}{g}$ where

w = weight

$g \approx 32.2$

a = acceleration (Feet/sec/sec)

Find the force necessary to accelerate a 100 pound object 3 feet per sec. per sec; 10 feet per sec. per sec; 100 feet per sec. per sec.

239. The equation for the aspect ratio of an airplane wing is $AR = \frac{b^2}{s}$

Where AR = aspect ratio

b = wing span

s = wing area

If the desired aspect ratio is 10 and the wing area is 2000 square feet, find the span. The area remaining the same, find the span when the aspect ratio equals 8, when it equals 6.

240. The equation for calculating the lift from the wing of an airplane is as follows

$$L = C_l \frac{\rho V^2}{2}$$

Where L = lift in pounds

$\rho = 0.02378$ = a function of the density of air

$S = 200$ = wing area in square feet

$C_l = 5$ = the lift coefficient of the air foil section selected for the particular wing, usually determined by wind tunnel tests

V = speed of plane in feet per sec

Find L when $V = 100, 150, 200$.

241. The T H P (thrust horsepower) delivered through a propeller = $BHP \times N$
Where BHP = brake horse power of the engine.

N = efficiency of the propeller

Let us assume that the efficiency of a certain propeller installed on an engine of 1200 B H P is 82%. Plot the curve of thrust horsepower versus brake horsepower throughout the brake horsepower range of the engine.

242. Let us assume that the brake horsepower of the engine mentioned in the preceding problem varies directly with the R.P.M. (revolutions per minute) and that the engine develops 1200 B H P @ 2,000 R.P.M. Plot a curve of thrust horsepower versus R.P.M. of the engine.

GEOMETRY — PROBLEMS

Sexagesimal Measurement

Solve the following:

1. $(25^\circ 13' 14'') + (12^\circ 4' 51'') =$
2. $(2^\circ 33' 43'') + (158^\circ 43' 18'') =$
3. $(23^\circ 3' 4'') - (13^\circ 43' 7'') =$
4. $(88^\circ 43' 12'') - (48^\circ 52' 13'') =$
5. $2(4^\circ 13' 8'') =$
6. $5(33^\circ 28' 48'') =$
7. $3(48^\circ 58' 23'') =$
8. $\frac{14^\circ 34' 26''}{2} =$
9. $\frac{38^\circ 14' 28''}{3} =$
10. $\frac{342^\circ 58' 7''}{8} =$

Natural Measurement

11. The length of an arc of a circle is $17''$ and the diameter is $34''$. What is the angle in radians that is subtended by the arc.
12. An angle of π radians is subtended by an arc of a circle with a radius of $20''$. Find the length of the arc.
13. $\theta = 4$ radians subtended by an $8''$ arc. Find the diameter of the circle represented by this arc.
14. Prove that a complete circle subtends an angle of 2π radians. Hint: Let perimeter equal \propto (length of the subtending arc).
15. Solve the following:

(a) 14° = radians	(d) 2 radians = $?^\circ$
(b) 90° = radians	(e) $.5\pi$ radians = $?^\circ$
(c) 150° = radians	(f) 4π radians = $?^\circ$
16. The radius of a circle is $20''$. How many degrees of the circle does a $20''$ arc subtend.
17. The angle subtended by a $15''$ arc is equal to 60° . What is the diameter of the circle?
18. Assuming that the diameter of the circle in the preceding problem was not known, find its perimeter.

TRIGONOMETRY — PROBLEMS

Types of Triangles

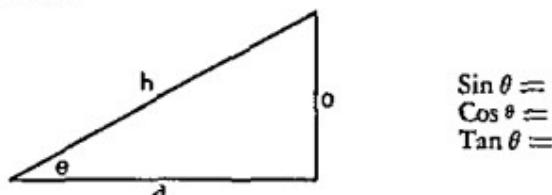
1. Name the two types of triangles.
2. What is a right triangle?

Elements of Triangles

3. Name the six elements of a triangle.
4. What are three ways of determining if a triangle is a right triangle?

Trigonometric Function in Right Triangles

5. Define the sine, cosine and tangent of the angle θ in the right triangle shown below.



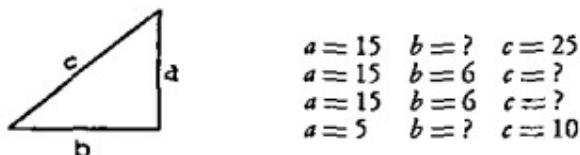
6. State the reciprocal functions for the Sine, Cosine and Tangent.

FUNCTION	RECIPROCAL FUNCTION
SIN	
COS	
TAN	

7. State the reciprocal functions of problem 6 in terms of a , o , and b as in Problem 5.

Geometric Relations

8. State mathematically the relationships between the three sides on a right triangle



10. If a triangle, similar to that in Problem 9 (a) has a side opposite equal to 12, what will be the hypotenuse?

Use of Tables of Natural Trigonometric Functions—Interpolation

11. Find the Sine of $10^\circ 10' 10''$
 12. Find the Cosine of $30^\circ 30' 30''$
 13. Find the Tangent of $50^\circ 50' 50''$
 14. Evaluate $\sin^{-1} .18665$

Solution of Right Triangles

15. Determine the value of the two acute angles in a right triangle with sides equal to 3-4-5.
 16. Determine the values of the Sine, Cosine and Tangent for each angle of Problem 15.
 17. Give from memory the values of the Sine, Cosine and Tangent of a 30° - 60° right triangle.
 18. Repeat Problem 17 for a 45° triangle.
 19. Using the Cotangent Functions, find the side adjacent of the 30° - 60° right triangles with the sides opposite the 30° angles equal to 55, 75 and 95.

20.

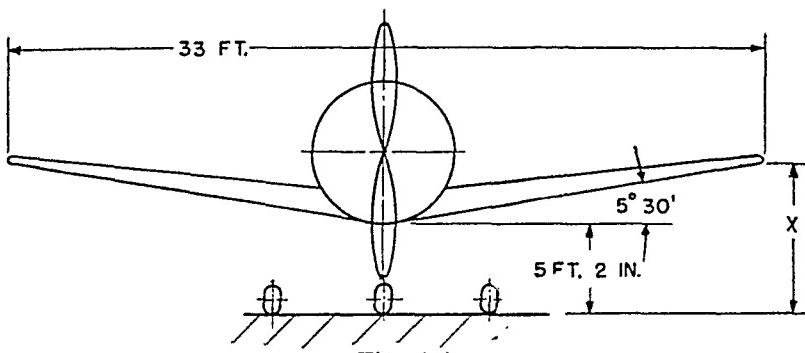


Fig. (a)

If an airplane as that shown in Fig. (a) has a wing spread of 33 feet, a ground clearance of 5 feet 2 inches and a airfoil angle of $5^{\circ}30'$, what is the distance from the ground to the wing tip?

Trigonometric Functions in Oblique Triangles

21. With the center of a circle of unit radius, located at the intersection of the ordinate and abscissa line indicate angles or 15° ; 108° ; 210° and 300° .
22. Indicate which quadrant each angle of Problem 21 is in.
23. Find the Sin, Cos, Tan, Cosec, Sec, and Cotan of each angle of Problem 21. Indicate correct Sign—plus or minus.

Geometric Representation of the Trigonometric Functions

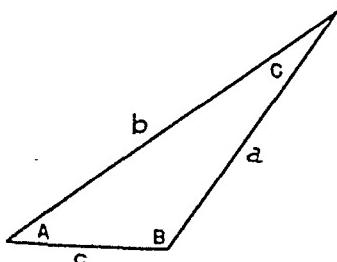
24. By use of a circle with a radius of unity, show diagrammatically the values of the Sin, Cos, Tan, Cotan, Sec, and Cosec.

Value of the Functions of Obtuse Angles

25. Find the Sin, Cos, and Tan of 100° , 200° and 280° and review the whole problem of functions of such angles.

Oblique Triangles Solved As Right Triangles

26. Solve the following problems:



	A°	B°	C°	a	b	c
a)	28	101	?	?	32	?
b)	21	153	?	57	?	?
c)	64	?	49	?	?	13
d)	?	?	?	39	72	51
e)	?	?	?	10	25	35
f)	16	?	?	?	36	18
g)	?	128	?	99	?	63

Oblique Triangles Solved By Special Formulas

27. Solve and check parts (a), (b), (f) and (g) of Problem 26 by the use of the Special Formulas which are noted under the above heading in the text

28. In reference to Solving of Oblique Triangles

 - (a) When can we use the Sine Proportions?
 - (b) When can we use the Cosine Law?
 - (c) When can we use the Law of Tangents?

29. Review all derivations of the Laws referred to in Problem 28.

Trigonometric Formulas

(Refer to Formulas 218-226 incl.)

- 30 Express the following in terms of $\sin \phi$ and $\cos \phi$ only
 (a) $\tan \phi + \sec \phi$ (d) $\cos \phi \tan \phi + \sec \phi$
 (b) $\sin^2 \phi + \sec^2 \phi$ (e) $\sin \phi \cos \phi \tan \phi \cot \phi$
 (c) $\cot^2 \phi + \csc^2 \phi$

31. Express the following in terms of $\sin \phi$ only.
 (a) $\sec^2 \phi + \tan^2 \phi$
 (b) $\sec \phi \tan \phi + \csc \phi$

32. Proof the following identities by transforming the left side only:
 (a) $\sin \phi \cot \phi = \cos \phi$ (d) $\tan \phi + \cot \phi = \sec \phi \csc \phi$
 (b) $(1 + \tan^2 \phi) \sin^2 \phi = \tan^2 \phi$ (e) $\sin \phi - \tan \phi = \tan \phi (\cos \phi - 1)$
 (c) $\frac{\sec^2 \phi - \tan^2 \phi}{\sec^2 \phi} = \csc^2 \phi$ (f) $\frac{\sin \phi - \cos \phi}{\sin \phi + \cos \phi} = \frac{\tan \phi - 1}{\tan \phi + 1}$

33. Solve the following by transposing both sides of the equation:
 (a) $\tan \phi + \cot \phi = \sec \phi \csc \phi$
 (b) $\sin \phi (1 + \tan \phi) - \sec \phi = \csc \phi - \cos \phi (1 + \cot \phi)$

34. (Refer to formulas 227-234 incl.)
 Prove the following:
 (a) $\sin(90^\circ + \phi) = \cos \phi$
 (b) $\cos(270^\circ - \phi) = -\sin \phi$

35. By use of Tables find approximate values of the following.
 (a) $\sin(47^\circ + 32^\circ) - (47^\circ + \sin 32^\circ)$
 (b) $\sin(25^\circ - 10^\circ) - (\sin 25^\circ - \sin 10^\circ)$

36. Find the exact values of the following, assuming that ϕ and β are positive acute angles
 (a) $\sin(\phi + \beta)$ if $\sin \phi = \frac{5}{13}$, $\cos B = \frac{4}{5}$
 (b) $\sin(\phi - \beta)$ if $\cos \phi = \frac{4}{5}$, $\cos \beta = \frac{5}{13}$
 (c) $\tan(\phi - \beta)$ if $\tan \phi = \frac{3}{4}$, $\sin \beta = \frac{15}{17}$

37. Solve the following identities
 (a) $\sin(60^\circ + \phi) = \frac{\sqrt{3} \cos \phi + \sin \phi}{2}$
 (b) $\cos(A - B) \cos B - \sin(A - B) \sin B = \cos A$
 (c) $\frac{\tan(x - y) + \tan y}{1 - \tan(x - y) \tan y} = \tan x$

38. Prove the formula for $\cos(\phi + \beta)$ directly from a figure in which ϕ and β are positive angles terminating the second quadrant and $\phi + \beta$ an angle terminating the third quadrant.
39. (Refer to formulas 235-244 incl.)
Find the exact values of $\sin 2\phi$; $\cos 2\phi$ and $\tan 2\phi$ in the following:
- (a) $\sin \phi = \frac{3}{5}$, if $0^\circ < \phi < 90^\circ$
 (b) $\operatorname{ctn} \phi = \frac{4}{3}$, if $360^\circ < \phi < 450^\circ$
 (c) $\sin \phi = -\frac{12}{13}$, if $-90^\circ < \phi < 0^\circ$
40. Find the exact values of the functions of the single angles of the following:
- (a) $\tan \phi = \frac{12}{5}$, if $180^\circ < \phi < 270^\circ$
 (b) $\sin \phi = -\frac{12}{13}$, if $-90^\circ < \phi < 0^\circ$
41. Prove the following identities:
- (a) $1 + \cos 2\phi = 2 \cos^2 \phi$
 (b) $\tan \phi = \frac{1 - \cos 2\phi}{\sin 2\phi} = \frac{\sin 2\phi}{1 + \cos 2\phi}$
 (c) $\cos^3 x - \sin^3 x = (\cos x - \sin x)(1 + \frac{1}{2} \sin 2x)$
 (d) $\sec 2\phi = 1 + \tan 2\phi \tan \phi$
 (e) $\cos 4\phi = 1 - 2 \sin^2 2\phi = 1 - 8 \sin^2 \phi \cos^2 \phi$
 (f) $\sin \phi = \frac{2 \tan \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}}$
42. (Refer to formulas 245-252 incl.)
Express the following as sums or differences of sines or of cosines.
- (a) $2 \sin 2\phi \cos 5\phi$
 (b) $\cos \phi \cos 2\phi$
 (c) $\sin 5\phi \cos 6\phi$
43. Express the following as products:
- (a) $\sin 40^\circ - \sin 50^\circ$
 (b) $\cos 20^\circ - \cos 30^\circ$
 (c) $\sin 3\phi - \sin \phi$
44. Express as a product involving only tangents and cotangents:
- (a) $\frac{\sin 40^\circ - \sin 20^\circ}{\sin 40^\circ + \sin 20^\circ}$
 (b) $\frac{\cos 2\phi - \cos \phi}{\cos 2\phi + \cos \phi}$

45. Prove the following identities:

$$(a) \sin 5x - \sin 3x = 2 \cos 4x \sin x$$

$$(b) \cos 8x + \cos 4x = 2 \cos 6x \cos 2x$$

$$(c) \frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}$$

$$(d) \sin x - \cos 2x - \sin 3x = -\cos 2x(2 \sin x + 1)$$

46. (Refer to eq. 253-255 incl.)

Solve for the values of angles A , B and C in a triangle with sides:
 $a = 6.20$; $b = 7.00$; $c = 9.10$

Area of Triangles

47. Find the area of the following triangles, using the parts given:

$$(a) b = 64.25, A = 24^\circ 23', B = 61^\circ 48'$$

$$(b) a = 549.2; c = 835; A = 41^\circ 8'$$

$$(c) a = 1653, b = 1777; c = 2131$$

LOGARITHMS — PROBLEMS

Introduction

1. What mathematical operations can be performed by logarithms?

Definitions and Principles

2. Applying the basic principles of *log* find the values of the following:
- (a) Multiply 3^8 by 3^2
 - (b) Multiply 10^2 by 10^3
 - (c) Divide 625 by 5^2
 - (d) Divide 10^6 by 10^5
 - (e) Raise 3^7 to the 6th power or $(3^7)^6 = ?$
 - (f) Raise 3^3 to the 3rd power or $(3^3)^3 = ?$
 - (g) Extract the square root of 81, or $\sqrt[3]{81} = ?$
 - (h) Express, with a fractional exponent, the cube root of 8 raised to the 7th power.
3. Define a *Logarithm*; and explain how the value of the *base* effects the logarithm.
4. What is a common *log*?
5. If the *log* of 1,000 is equal to 3.0000, which part of the *log* is the characteristic and which part is the mantissa?
6. (a) What part of the logarithm is recorded in *log* tables?
 (b) Are the values positive or negative?
7. Find the *log* of the following:
- | | |
|-----------|------------------------------|
| (a) .0002 | (f) 20. |
| (b) .002 | (g) 200. |
| (c) .02 | (h) 2,000 |
| (d) .2 | (i) 20,000 |
| (e) 20 | (Given that: Log 2.0 030103) |

8. Express the following in their equivalent form, such as $-2,301030 = \bar{3}.698970$ and prove mathematically why this is true.
- -2.33968
 - -9.84680
 - $\bar{4}.82827$
 - $\bar{1}.00657$

Rules for Characteristics

9. Review the two methods of determining the characteristic of a *log*.
 10. State the characteristic of each log in Problem 7 above.

Augmented Logarithms

11. What are augmented *logs*, and why are they used?
 12. Express all the parts of Problem 8 as augmented *logs* and prove mathematically why they are true.
 13. Find the values of x , y and z by augmented *logs* and prove by applying the basic principles of *logs*.

Use of Tables of Logarithms

14. Find the *log* of:
- | | |
|-----------|----------------|
| (a) 351 | (f) 0.222 |
| (b) 59.6 | (g) 0.0064 |
| (c) 9.99 | (h) 1,022. |
| (d) 1.44 | (i) 2,000,000 |
| (e) 0.749 | (j) 199.9 |
15. Find the numbers whose *logs* are:
- | | |
|---------------------|---------------------|
| (a) 3.00217 | (f) -1.77048 |
| (b) 2.87040 | (g) -6.52699 |
| (c) 0.90037 | (h) $\bar{3}.59040$ |
| (d) 7.64048 | (i) 1.95294 |
| (e) $\bar{1}.38021$ | (j) 2.99917 |
16. By applying the principles of interpolation, find the *logs* of the following:
- | | |
|--------------|---------------|
| (a) 0.029968 | (f) 7.76768 |
| (b) 2.11370 | (g) 0.00666 |
| (c) 33.00770 | (h) 2,294,600 |
| (d) 0.400680 | (i) 926,106 |
| (e) 1.19230 | (j) 2.66870 |
17. Find the numbers whose *logs* are:
- | | |
|---------------------|----------------------|
| (a) 2.27199 | (f) 7.65749 |
| (b) 1.09081 | (g) 4.97505 |
| (c) $\bar{2}.72333$ | (h) 0.01498 |
| (d) $\bar{4}.83018$ | (i) 0.30227 |
| (e) $\bar{1}.70779$ | (j) $\bar{1}.47425$ |

Fundamental Operations Using Logarithms

18. Find the values of the following by *logs*:
- | | |
|-------------------|------------------------|
| (a) (10) (760) | (f) (2.88) (7.98) |
| (b) (105) (88) | (g) (1.14) (2,000,800) |
| (c) (8) (967) | (h) (0.008) (0.00676) |
| (d) (291) (298) | (i) (0.186) (22.83) |
| (e) (0.77) (0.88) | (j) (3.43) (6,845) |

- 19 (a) $92/87$ (f) $100/2,100$
 (b) $11/21$ (g) $2/606$
 (c) $0.6/0.47$ (h) $0.884/0.821$
 (d) $2,117/124$ (i) $1.835/2,647$
 (e) $899/62$ (j) $1,008,000/67,200$
- 20 (a) 25^4 (f) 88^6
 (b) 6^6 (g) $7,620^2$
 (c) 294^2 (h) $(0.11)^3$
 (d) 4^{21} (i) $(0.002)^3$
 (e) 968^1 (j) 3.8^5
- 21 (a) $\sqrt{84}$ (f) $\sqrt{272}$
 (b) $\sqrt[3]{6,942}$ (g) $\sqrt[3]{88}$
 (c) $\sqrt[3]{1,010}$ (h) $\sqrt[3]{675}$
 (d) $\sqrt[3]{3,260}$ (i) $\sqrt[3]{7}$
 (e) $\sqrt[3]{0.1186}$ (j) $\sqrt[3]{1000}$

Cologarithms

- 22 What is a *cologarithm*?
 23 Solve all of Problem 19 by the use of *cologs*.

Division or Multiplication of Logarithms

- 24 Is it true that the difference of the *logs* of two numbers is equal to the *log* of the difference of the two numbers? Prove.
- 25 Solve for (x) in the following
 (a) $(x) (\log 299) = \log 68.4$
 (b) $(x) (\log 41) = \log 8,960$
 (c) $x = (\log 7,600) (\log 6)$
 (d) $\frac{96}{213} = (\frac{1}{2})^x$

Solution of Equations Using Logarithms

- 26 Solve for x
 (a) $(9)^{1/6} = (266)(x)$
 (b) $187.8 = (x)(18)^3$
 (c) $(23.7)^{3/3} = (22)^{1/4}(x)$
27. Solve for x
 (a) $x = \sqrt{\frac{2,120}{(100)^2(227.6)}}$ (c) $(624.0)^2(x) = \sqrt[3]{\frac{2,000,000}{(877)(1.022)}}$
 (b) $(89.7)(x) = \sqrt[3]{39.86}$ (d) $x = \sqrt{\frac{1,298}{(400)^2(34.7)}}$
28. Solve for x
 (a) $x = \left[\frac{3,768}{(.877)^3} \right]^{5/3}$
 (b) $x = \left[\frac{.00876}{.01273} \right]^{2/3}$
 (c) $107.6x = [01080]^{6/12}$

29. Solve for x

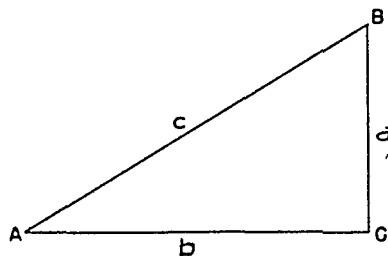
$$(a) \quad x = \left[122 - \left(\frac{2}{6.8} \right)^{.8} \right] 28.8$$

$$(b) \quad (3.7)^2 x = \left[8.7 - \left(\frac{1}{4.4} \right)^{.23} \right] 37.88$$

$$(c) \quad x = \left[1 - \left(\frac{1}{11} \right)^{.011} \right] 2,677$$

Solution of Triangles Using Logarithms

30. Find the unknowns of the following triangles by natural logs and logs of trigonometric functions:



$$(a) \quad c = 120; A = 31^\circ$$

$$(b) \quad a = 637; A = 4^\circ 35'$$

$$(c) \quad a = 2.189; B = 45^\circ 25'$$

$$(d) \quad b = 16.93; B = 51^\circ 2'$$

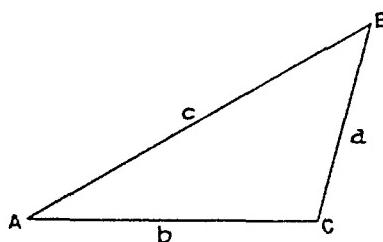
$$(e) \quad a = 0.7183; c = 9.914$$

31. (a) $A = 47^\circ 13'; B = 65^\circ 24'; a = 43.18$

$$(b) \quad A = 65^\circ 50'; B = 38^\circ 37'; b = 835.6$$

$$(c) \quad A = 68^\circ 41'; B = 1^\circ 2'; c = 9.433$$

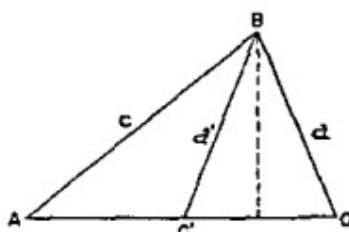
$$(d) \quad A = 61^\circ 27'; C = 33^\circ 22'; a = 3541$$



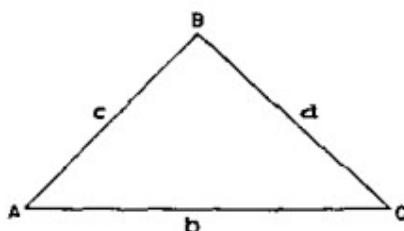
32. (a) $a = 443; c = 439; A = 40^\circ 12'$

$$(b) \quad a = 724.7; c = 787.5; A = 65^\circ 15'$$

- (c) $a = 1149, b = 1246; A = 67^\circ 16'$
 (d) $a = 3541; c = 4017; A = 61^\circ 27'$



- 33 (a) $a = 7, b = 9, C = 48^\circ$ Find c
 (b) $b = 100, c = 200, A = 29^\circ 34'$ Find a
 (c) $a = 1471, b = 1759, C = 43^\circ 43'$
 (d) $b = 9641, c = 8999, A = 67^\circ 21'$
 (e) $b = 2525, c = 9746, A = 98^\circ 49'$



- 34 Write the equations for converting common *logs* to natural *logs* and visa versa
 35 If the common *logs* of x are as follows, find the natural *log* equivalents:
 (a) $\text{Log}_{10} X = 0.8186$
 (b) $\text{Log}_{10} X = 1.8912$
 (c) $\text{Log}_{10} X = 0.00113$
 36 If the natural *logs* are given, find the common *logs*.
 (a) $\text{Log}_e X = 0.6931$
 (b) $\text{Log}_e X = 4.7281$
 (c) $\text{Log}_e X = 1.4310$

Natural or Napierian Logarithms

- 37 Find the natural *logs* of the following
 (a) 0.0097 (d) 0.18431
 (b) 1.68270 (e) 400,000
 (c) 102,870

ANALYTICAL GEOMETRY OF STRAIGHT LINES — PROBLEMS

Introduction

1. Define (a) Dependent Variable and
 (b) Independent Variable

Give examples

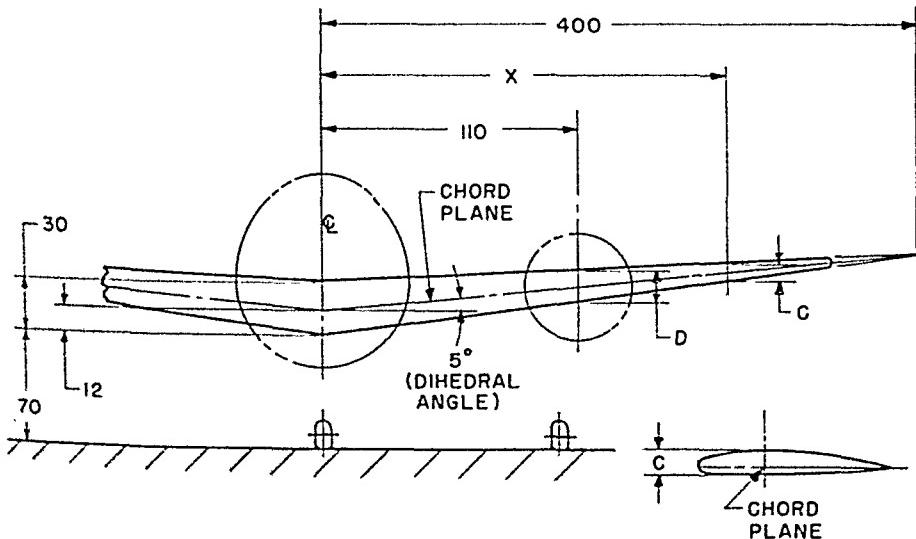
Straight Lines

2. What are the distinctive characteristics of the lines $y = 6$; $y = 22$ and $x = 10$.
3. Does the line $y = 99x$ pass through the ordinate? If not what is the intercept?
4. Plot the curve of Problem 3.
5. May the equation of the line of Problem 3 be classed as a linear equation? If so, why?
6. Does the point $x = 13$; $y = 1287$; lie on the curve of Problem 3? Prove.

Slope of a Straight Line

7. Define positive and negative slopes.
8. When does a line have zero slope?
9. When does a line have no slope?
10. If (x) decreases and (y) also decreases, what is the sign of the slope of the line?
11. If (y) decreases and (x) increases, what is the sign of the slope of the line?
12. What are the axes' intercepts of the following equations:

(a) $y = 2x + 4$	(c) $3y = -4x + 8$
(b) $y = 2x - 20$	(d) $4y = 4x + 4$
13. What are the values and sign values of the slopes of the curves of the four curves of Problem 12?
14. From Fig. below determine the equation which expresses the relationship between x and C . Using this equation find the value of D .

**Interpretation of Slope Term**

15. What kind of lines have their slopes equal?
16. If the slope of a line is equal to the negative reciprocal of that of another, what is the relationship between the two lines?

SOLUTION OF EQUATIONS

- 17 If one number is the negative reciprocal of another, is their product or their quotient equal to minus one?
- 18 Prove which of the following pairs of lines are parallel, perpendicular, or neither:
- | | |
|------------------------------------|-----------------------------------|
| (a) $y = 2x + 4$ | (c) $y = 12x$ |
| (a ₁) $6y = -10 + 12x$ | (c ₁) $3y = -36x + 7$ |
| (b) $2x = 10 + y$ | (d) $14x = 18y - 6$ |
| (b ₁) $3y - 6x = 2$ | (d ₁) $100 + 4y = 3x$ |
- 19 Find the angle between the two lines of Problem 18 which are neither parallel nor perpendicular.
- 20 If the slope of one line is equal to -68 and that of a second line equals $.82$, what is the angle between the two in degrees and minutes?

Equation of Any Straight Line

- 21 Write the equations of the following lines if the slopes and the intercepts are as follows:

Slope	Intercept
(a) +2	$x = -2, y = 0$
(b) +7	$x = +4; y = 0$
(c) -66	$x = 0, y = 0$
(d) +1	$y = -22, x = 0$
(e) -2	$y = -11, x = 0$

Such as $y = (m)(x) + b$

- 22 Write the equations of the following lines if the slopes of the lines and a point on each line are as follows:

Slope	Point (P)
(a) +9	$x = 0, y = 2$
(b) -2	$x = 9; y = 3$
(c) +2	$x = -2; y = +2$
(d) -4	$x = 12, y = 1$
(e) +1	$x = -4; y = -6$

- 23 Is point P_2 ($x = 18, y = 6$) on the same line as that of part (b) in Problem 22 above?

Simultaneous Equation of Straight Lines

- 24 Is it true that a set of lines must intersect at but one point to fulfill the requirements of independent equations?
- 25 Define inconsistent and also equivalent equations?
- 26 Which set of lines of Problem 18 are independent, inconsistent or equivalent?

Distance from a Point to a Straight Line

- 27 What is the shortest distance from a point to a line?
- 28 Find the shortest distances between the following points and their respective lines:
- | | |
|----------------------------------|-------------------------|
| (a) Point. $x = 10, y = 2$, | Line $4x - 4y + 15 = 0$ |
| (b) Point. $x = 1, y = 20$, | Line $9x + y = 0$ |
| (c) Point $x = 5, y = 8$, | Line $3x + 8y = 17$ |

Area Beneath a Straight Line Segment

29. Find the area under the curve $y = 9x + 10$ which is between the two vertical lines drawn at $x = 1$ and $x = 2$ and the horizontal line $y = 0$.
30. Repeat the above for:
 (a) Line..... $2x + 9y - 6 = 0$
 $x = 0$
 $x = 9$
 $y = 0$
 (b) Line..... $7x + 10y + 10 = 0$
 $x = 3$
 $x = 4$
 $y = 2$

APPENDIX — SLIDE RULE — PROBLEMS**Division**

- | | | |
|-----------------|-----------------------|--------------------------|
| 1. $8 \div 32$ | 8. $450 \div 1500$ | 15. $18.7 \div 0.214$ |
| 2. $32 \div 8$ | 9. $12.4 \div 13.8$ | 16. $.00626 \div .00038$ |
| 3. $45 \div 82$ | 10. $7.38 \div 3.30$ | 17. $.00158 \div 37.6$ |
| 4. $82 \div 45$ | 11. $5.86 \div 2.11$ | 18. $.375 \div .015$ |
| 5. $61 \div 69$ | 12. $6.13 \div 4.27$ | 19. $.015 \div .375$ |
| 6. $80 \div 15$ | 13. $3.14 \div 10.00$ | 20. $.0475 \div 22.1$ |
| 7. $15 \div 80$ | 14. $10.00 \div 3.14$ | |

Multiplication

- | | | |
|----------------------|-------------------------|--------------------------|
| 21. 2×1.7 | 26. $3 \times .25$ | 31. 1.8×430 |
| 22. 1.3×3.1 | 27. $3 \times .025$ | 32. $.0018 \times 59.6$ |
| 23. 30×25 | 28. $.3 \times 25$ | 33. $.0782 \times .0789$ |
| 24. 300×25 | 29. $.03 \times .025$ | 34. $1,010 \times 2.66$ |
| 25. 30×250 | 30. 18.00×4.30 | |

35.

	16	17	18	19	20	21	22	23	24
40									
41									
42									
43									

Combined Multiplication and Division

- | | | |
|----------------------------------|----------------------------------|--|
| 36. $\frac{2 \times 3.6}{4}$ | 40. $\frac{11 \times 392}{36}$ | 43. $\frac{8.77 \times 87.2}{76.0 \ 3.1}$ |
| 37. $\frac{11 \times 16}{3.4}$ | 41. $\frac{8.2}{10.2 \times 17}$ | 44. $\frac{1.35 \times 4.71}{0.77 \times 8.1}$ |
| 38. $\frac{6 \times 7.8}{4.63}$ | 42. $\frac{55}{17.8 \ 2.8}$ | 45. $\frac{2,010 \times 399}{834}$ |
| 39. $\frac{100 \times 89}{2.87}$ | | |

SOLUTION OF EQUATIONS

Proportions

Find x for the following:

46.	$\frac{8}{1.2} = \frac{x}{7.6}$	50.	$\frac{100}{87.3} = \frac{53.4}{x}$	54.	$\frac{x}{12.4} = \frac{7.77}{87.6}$
47.	$\frac{1}{7.8} = \frac{x}{822}$	51.	$\frac{398}{412} = \frac{21}{x}$	55.	$\frac{33.3}{x} = \frac{5.30}{11.8}$
48.	$\frac{x}{5.31} = \frac{70.2}{4.07}$	52.	$\frac{2.85}{4.33} = \frac{x}{8}$	56.	$\frac{.0068}{2.63} = \frac{x}{.0194}$
49.	$\frac{287}{x} = \frac{1.88}{90.1}$	53.	$\frac{x}{2} = \frac{8.2}{11.2}$		

Square Roots and Squares of Numbers

Find the square root of the following

57.	.000218	61.	43.7	65.	834,000
58.	.218	62.	0187	66.	6,620
59.	21.8	63.	00290	67.	376
60.	8.77	64.	9.29		

Find the squares of the following

68.	311	73.	3.99	78.	.000722
69.	896	74.	11.7	79.	1020
70.	100	75.	462	80.	49.9
71.	621	76.	00411	81.	773
72.	178	77.	.285	82.	543

Square Root of the Sum or the Difference of Two Squares

83.	$\sqrt{30^2 + 120^2}$	85.	$\sqrt{9^2 + 101^2}$	87.	$\sqrt{100^2 - 10^2}$
84.	$\sqrt{45^2 + 81^2}$	86.	$\sqrt{80^2 - 22^2}$		

Cube Roots and Cubes of Numbers

Find the cube roots of the following

88.	20	92.	343	95.	.0000229
89.	118	93.	217	96.	6710
90.	707	94.	0178	97.	1000
91.	2010				

Find the cubes of the following

98.	11.1	102.	500	105.	1000
99.	31.0	103.	813	106.	199
100.	47.0	104.	2.89	107.	877
101.	480				

Trigonometric Functions

Find the Sine of the following:

108.	90°	112.	$1^\circ 6'$	115.	$10'$
109.	45°	113.	$8^\circ 31'$	116.	$71^\circ 30'$
110.	41°	114.	$99^\circ 30'$	117.	160°
111.	$10^\circ 32'$				

Find the Cosine of the following:

$$118. \quad 80^\circ$$

$$119. \quad 75^\circ$$

$$120. \quad 13^\circ$$

$$121. \quad 1^\circ 2'$$

$$122. \quad 54^\circ 15'$$

$$123. \quad 20^\circ 30'$$

$$124. \quad 87^\circ 45'$$

By the use of the slide rule find the *logs* of the following:

$$125. \quad 1.44$$

$$126. \quad 53.4$$

$$127. \quad 0.317$$

$$128. \quad 0.0722$$

$$129. \quad 443.0$$

Logarithms

By the use of the *log* scales find the value of the following:

$$130. \quad (3.3)^4$$

$$131. \quad (99)^5$$

$$132. \quad (10.2)^{10.1}$$

$$133. \quad \sqrt[3]{8.76}$$

$$134. \quad \sqrt[3]{213}$$

$$135. \quad \sqrt[1.2]{43.4}$$

Section VIII

TRIGONOMETRIC FUNCTIONS

0° READ DOWN 1°

	sin	tan	cot	cos			sin	tan	cot	cos	
	cos	cot	tan	sin	/		cos	cot	tan	sin	/
0	.00000	.00000	∞	1.0000	60	0	.01745	.01746	57.290	.99985	60
1	.029	.029	3437.7	.000	59	1	.774	.775	56.351	.984	59
2	.058	.058	1718.9	.000	58	2	.803	.804	55.442	.984	58
3	.087	.087	1145.9	.000	57	3	.832	.833	54.561	.983	57
4	.116	.116	859.41	.000	56	4	.862	.862	53.709	.983	56
5	.145	.145	687.55	.000	55	5	.891	.891	52.882	.982	55
6	.175	.175	572.96	.000	54	6	.920	.920	52.081	.982	54
7	.204	.204	491.11	.000	53	7	.949	.949	51.303	.981	53
8	.00233	.00233	429.72	.000	52	8	.01978	.01978	50.549	.980	52
9	.262	.262	381.97	.000	51	9	.02007	.02007	49.816	.980	51
10	.291	.291	343.77	1.0000	50	10	.036	.036	49.104	.99979	50
11	.320	.320	312.52	.99999	49	11	.065	.066	48.412	.979	49
12	.349	.349	286.48	.999	48	12	.094	.095	47.740	.978	48
13	.378	.378	264.44	.999	47	13	.123	.124	47.085	.977	47
14	.407	.407	245.55	.999	46	14	.152	.153	46.449	.977	46
15	.436	.436	229.18	.999	45	15	.181	.182	45.829	.976	45
16	.465	.465	214.86	.999	44	16	.211	.211	45.226	.976	44
17	.00495	.00495	202.22	.999	43	17	.240	.02240	44.639	.975	43
18	.524	.524	190.98	.999	42	18	.269	.269	44.066	.974	42
19	.553	.553	180.93	.998	41	19	.298	.298	43.508	.974	41
20	.582	.582	171.59	.99998	40	20	.02327	.328	42.964	.89973	40
21	.611	.611	163.70	.998	39	21	.356	.357	42.433	.972	39
22	.640	.640	156.26	.998	38	22	.385	.386	41.916	.972	38
23	.669	.669	149.47	.998	37	23	.414	.415	41.411	.971	37
24	.698	.698	143.24	.998	36	24	.443	.444	40.917	.970	36
25	.00727	.00727	137.51	.997	35	25	.472	.473	40.436	.969	35
26	.756	.756	132.22	.997	34	26	.501	.02502	39.965	.969	34
27	.785	.785	127.32	.997	33	27	.530	.531	39.506	.968	33
28	.814	.815	122.77	.997	32	28	.560	.560	39.057	.967	32
29	.844	.844	118.54	.996	31	29	.589	.589	38.618	.966	31
30	.873	.873	114.50	.99999	30	30	.02618	.619	38.188	.09966	30
31	.902	.902	110.89	.996	29	31	.647	.648	37.769	.965	29
32	.931	.931	107.43	.996	28	32	.676	.677	37.358	.964	28
33	.960	.960	104.17	.995	27	33	.705	.706	36.956	.963	27
34	.00989	.00989	101.11	.995	26	34	.734	.735	36.563	.963	26
35	.01018	.01018	98.218	.995	25	35	.763	.02764	36.178	.962	25
36	.047	.047	95.459	.995	24	36	.792	.793	35.801	.961	24
37	.076	.076	92.905	.994	23	37	.822	.02821	35.431	.960	23
38	.105	.105	90.463	.994	22	38	.850	.851	35.070	.959	22
39	.134	.135	88.144	.994	21	39	.879	.881	34.715	.959	21
40	.164	.164	85.910	.99993	20	40	.908	.910	34.368	.99958	20
41	.193	.193	83.814	.993	19	41	.938	.939	34.027	.957	19
42	.222	.222	81.847	.993	18	42	.967	.968	33.694	.956	18
43	.01251	.251	79.943	.992	17	43	.02996	.02997	33.366	.955	17
44	.280	.01280	78.126	.992	16	44	.03025	.03026	33.045	.954	16
45	.309	.309	76.390	.991	15	45	.054	.055	32.730	.953	15
46	.338	.338	74.729	.991	14	46	.083	.084	32.421	.952	14
47	.367	.367	73.139	.991	13	47	.112	.114	32.118	.952	13
48	.396	.396	71.615	.990	12	48	.141	.143	31.821	.951	12
49	.425	.425	70.153	.990	11	49	.170	.172	31.528	.950	11
50	.454	.455	68.750	.9999	10	50	.199	.201	31.242	.09949	10
51	.483	.484	67.402	.999	9	51	.228	.230	30.960	.948	9
52	.01513	.01513	66.105	.998	8	52	.03257	.03259	30.683	.947	8
53	.542	.542	64.855	.998	7	53	.286	.288	30.412	.946	7
54	.571	.571	63.657	.998	6	54	.316	.317	30.145	.945	6
55	.600	.600	62.499	.997	5	55	.345	.346	29.882	.944	5
56	.629	.629	61.383	.997	4	56	.374	.376	29.624	.943	4
57	.658	.658	60.306	.996	3	57	.403	.405	29.371	.942	3
58	.687	.687	59.266	.996	2	58	.432	.434	29.122	.941	2
59	.716	.716	58.261	.995	1	59	.461	.463	28.877	.940	1
60	.01745	.01746	57.290	.99985	0	60	.03490	.03492	28.636	.99939	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

89°

READ UP

89°

TRIGONOMETRIC FUNCTIONS

251

READ DOWN						3°					
'	sin	tan	cot	cos		'	sin	tan	cot	cos	
0	.03490	.03492	28. 636	.99939	60	0	.05234	.05241	19.081	.99863	60
1	519	521	.399	938	59	1	263	270	18.976	861	59
2	548	550	28.166	937	58	2	292	299	.871	860	58
3	577	579	27.937	936	57	3	321	328	.768	858	57
4	606	609	.712	935	56	4	350	357	.666	857	56
5	635	638	.490	934	55	5	379	387	.564	855	55
6	664	667	.271	933	54	6	408	416	.464	854	54
7	693	696	27.057	932	53	7	437	445	.366	852	53
8	723	.03752	26.845	931	52	8	466	474	.268	851	52
9	.03752	754	.637	930	51	9	.05495	.05503	.171	849	51
10	781	783	.432	.99929	50	10	524	533	18.075	.99847	50
11	810	812	.230	927	49	11	553	562	17.980	846	49
12	839	842	26.031	926	48	12	582	591	.886	844	48
13	868	871	25.835	925	47	13	611	620	.793	842	47
14	897	900	.642	924	46	14	640	649	.702	841	46
15	926	929	.452	923	45	15	669	678	.611	839	45
16	955	958	.264	922	44	16	698	708	.521	838	44
17	.03984	.03987	25.080	921	43	17	.05727	737	.431	836	43
18	.04013	.04016	24.898	919	42	18	756	.05766	.343	834	42
19	042	046	.719	918	41	19	785	795	.256	833	41
20	071	075	.542	.99917	40	20	814	824	.169	.99831	40
21	100	104	.368	916	39	21	844	854	17.084	829	39
22	129	133	.196	915	38	22	873	883	16.999	827	38
23	159	162	24.026	913	37	23	902	912	.915	826	37
24	188	191	23.859	912	36	24	931	941	.832	824	36
25	217	220	.695	911	35	25	960	970	.750	822	35
26	246	.04250	.532	910	34	26	.05989	.05999	.668	821	34
27	.04275	279	.372	909	33	27	.06018	.06029	.587	819	33
28	304	308	.214	907	32	28	047	058	16.507	817	32
29	333	337	23.058	906	31	29	076	087	.428	815	31
30	362	366	22.904	.99905	30	30	105	116	.350	.99813	30
31	391	395	.752	904	29	31	134	145	.272	812	29
32	420	424	.602	902	28	32	163	175	.195	810	28
33	449	454	.454	901	27	33	192	204	.119	808	27
34	478	.04483	.308	900	26	34	221	233	16.043	806	26
35	.04507	512	.164	898	25	35	.06250	262	15.969	804	25
36	536	541	22.022	897	24	36	279	.06291	.895	803	24
37	565	570	21.881	896	23	37	308	321	.821	801	23
38	594	599	.743	894	22	38	337	350	.748	799	22
39	623	628	.606	893	21	39	366	379	.676	797	21
40	653	658	.470	.99892	20	40	395	408	.605	.99795	20
41	682	687	.337	890	19	41	424	438	.534	793	19
42	.04711	.04716	.205	889	18	42	453	467	15.464	792	18
43	740	745	21.075	888	17	43	.06482	496	.394	790	17
44	769	774	20.946	886	16	44	511	.06525	.325	788	16
45	798	803	.819	885	15	45	540	554	.257	786	15
46	827	833	.693	883	14	46	569	584	.189	784	14
47	856	862	.569	882	13	47	598	613	.122	782	13
48	885	891	.446	881	12	48	627	642	15.056	780	12
49	914	920	.325	879	11	49	656	671	14.990	778	11
50	943	949	.206	.99878	10	50	685	700	.924	.99776	10
51	.04972	.04978	20.087	876	9	51	.06714	730	.860	774	9
52	.05001	.05007	19.970	875	8	52	743	.06759	.795	772	8
53	030	037	.855	873	7	53	773	788	.732	770	7
54	059	066	.740	872	6	54	802	817	.660	768	6
55	088	095	.627	870	5	55	831	847	.606	766	5
56	117	124	.516	869	4	56	860	876	.544	764	4
57	146	153	.405	867	3	57	889	905	.482	762	3
58	175	182	.296	866	2	58	918	934	.421	760	2
59	205	212	.188	864	1	59	947	963	.361	758	1
60	.05234	.05241	19.081	.99863	0	60	.06976	.06993	14.301	.99756	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'

87°

READ UP

86°

SOLUTION OF EQUATIONS

4° READ DOWN

5°

/	sin	tan	cot	cos	/	sin	tan	cot	cos		
0	.06976	.06993	14.301	.99756	60	0	.08716	.08749	11.430	.99819	60
1	.07005	.07022	.241	.754	59	1	.745	.778	.392	.617	59
2	.034	.051	.182	.752	58	2	.774	.807	.354	.614	58
3	.063	.080	.124	.750	57	3	.803	.837	.316	.612	57
4	.092	.110	.065	.748	56	4	.831	.866	.279	.609	56
5	.121	.139	14.008	.746	55	5	.860	.895	.242	.607	55
6	.150	.168	13.951	.744	54	6	.889	.925	.205	.604	54
7	.179	.197	.894	.742	53	7	.918	.954	.168	.602	53
8	.208	.227	.838	.740	52	8	.947	.08983	.132	.599	52
9	.237	.07256	.782	.738	51	9	.08976	.09013	.095	.596	51
10	.07266	.235	.727	.99736	50	10	.09005	.042	.050	.99594	50
11	.295	.314	.672	.734	49	11	.034	.071	11.024	.581	49
12	.324	.344	.617	.731	48	12	.063	.101	10.988	.588	48
13	.353	.373	.563	.729	47	13	.092	.130	.953	.586	47
14	.382	.402	13.510	.727	46	14	.121	.159	.918	.583	46
15	.411	.431	.457	.725	45	15	.150	.189	.893	.580	45
16	.440	.461	.404	.723	44	16	.179	.218	.818	.578	44
17	.469	.490	.352	.721	43	17	.208	.09247	.814	.575	43
18	.07498	.07519	.300	.719	42	18	.09237	.277	.780	.572	42
19	.527	.548	.248	.715	41	19	.266	.306	.748	.570	41
20	.556	.578	.197	.99714	40	20	.295	.335	.712	.99567	40
21	.585	.607	.146	.712	39	21	.324	.365	.678	.561	39
22	.614	.636	.096	.710	38	22	.353	.394	10.645	.562	38
23	.643	.665	13.016	.708	37	23	.382	.423	.612	.559	37
24	.672	.695	12.996	.705	36	24	.411	.453	.579	.556	36
25	.701	.724	.947	.703	35	25	.440	.09182	.546	.553	35
26	.07730	.07753	.898	.701	34	26	.09469	.511	.514	.551	34
27	.759	.782	.850	.699	33	27	.498	.541	.481	.548	33
28	.788	.812	.801	.696	32	28	.527	.570	.449	.545	32
29	.817	.841	.754	.694	31	29	.556	.600	.417	.542	31
30	.846	.870	.706	.99692	30	30	.585	.629	.385	.99540	30
31	.875	.899	.659	.689	29	31	.614	.658	.354	.537	29
32	.904	.929	12.612	.687	28	32	.642	.688	10.322	.534	28
33	.933	.958	.566	.685	27	33	.671	.09717	.291	.531	27
34	.962	.07937	.520	.683	26	34	.700	.746	.260	.528	26
35	.07991	.08017	.474	.680	25	35	.09729	.776	.229	.526	25
36	.08020	.046	.429	.678	24	36	.758	.805	.199	.523	24
37	.049	.075	.381	.676	23	37	.787	.834	.168	.520	23
38	.078	.104	12.330	.673	22	38	.816	.864	.138	.517	22
39	.107	.134	.295	.671	21	39	.845	.893	.108	.514	21
40	.136	.163	.251	.99658	20	40	.874	.923	.078	.99511	20
41	.165	.192	.207	.666	19	41	.903	.952	.048	.508	19
42	.194	.221	.163	.664	18	42	.932	.09981	10.019	.506	18
43	.223	.08251	.120	.661	17	43	.961	.10011	9.9893	.503	17
44	.08252	.280	.077	.659	16	44	.09090	.040	.9601	.500	16
45	.281	.309	12.035	.657	15	45	.10019	.069	.9310	.497	15
46	.310	.339	11.992	.654	14	46	.048	.099	.9021	.494	14
47	.339	.368	.950	.652	13	47	.077	.128	.8734	.491	13
48	.368	.397	.909	.649	12	48	.106	.158	.8448	.488	12
49	.397	.427	.867	.647	11	49	.135	.187	.8164	.485	11
50	.426	.456	.826	.99644	10	50	.164	.216	.7882	.99182	10
51	.455	.485	.785	.642	9	51	.192	.10246	.7601	.479	9
52	.08184	.08514	.745	.639	8	52	.221	.275	.9.7322	.476	8
53	.513	.544	.705	.637	7	53	.10250	.305	.7044	.473	7
54	.542	.573	11.664	.635	6	54	.279	.334	.6766	.470	6
55	.571	.602	.625	.632	5	55	.308	.363	.6493	.467	5
56	.600	.632	.585	.630	4	56	.337	.393	.6220	.464	4
57	.629	.661	.546	.627	3	57	.366	.422	.5949	.461	3
58	.658	.690	.507	.625	2	58	.395	.452	.5679	.456	2
59	.687	.720	.468	.622	1	59	.424	.481	.5411	.453	1
60	.08716	.08749	11.430	.99619	0	60	.10153	.10510	9.5144	.99452	0
	cos	cot	tan	sin			cos	cot	tan	sin	

85°

READ UP

84°

6° READ DOWN 7°

	sin	tan	cot	cos			sin	tan	cot	cos	
0	.10453	.10510	9.5144	.99452	60	0	.12187	.12278	8.1443	.99255	60
1	.482	.540	.4878	.449	59	1	.216	.308	.1248	.251	59
2	.511	.569	.4614	.446	58	2	.245	.338	.1054	.248	58
3	.540	.599	.4352	.443	57	3	.274	.367	.0860	.244	57
4	.569	.628	.4090	.440	56	4	.302	.397	.0667	.240	56
5	.597	.657	.3831	.437	55	5	.331	.426	.0476	.237	55
6	.626	.687	.3572	.434	54	6	.360	.456	.0285	.233	54
7	.655	.716	.3315	.431	53	7	.389	.485	8.0095	.230	53
8	.684	.10746	.3060	.428	52	8	.418	.12515	7.9906	.226	52
9	.10713	.775	.2806	.424	51	9	.12447	.544	.9718	.222	51
10	.742	.805	9.2553	.99421	50	10	.476	.574	.9530	.99219	50
11	.771	.834	.2302	.418	49	11	.504	.603	.9344	.215	49
12	.800	.863	.2052	.415	48	12	.533	.633	.9158	.211	48
13	.829	.893	.1803	.412	47	13	.562	.662	.8973	.208	47
14	.858	.922	.1555	.409	46	14	.591	.692	.8789	.204	46
15	.887	.952	.1309	.406	45	15	.620	.722	.8606	.200	45
16	.916	.10981	.1065	.402	44	16	.649	.12751	7.8424	.197	44
17	.945	.11011	.0821	.399	43	17	.678	.781	.8243	.193	43
18	.10973	.040	.0579	.396	42	18	.12706	.810	.8062	.189	42
19	.11002	.070	.0338	.393	41	19	.735	.840	.7882	.186	41
20	.031	.099	9.0098	.99390	40	20	.764	.869	.7704	.99182	40
21	.060	.128	.9860	.386	39	21	.793	.899	.7525	.178	39
22	.089	.158	.9623	.383	38	22	.822	.929	.7348	.175	38
23	.118	.187	.9387	.380	37	23	.851	.958	.7171	.171	37
24	.147	.217	.9152	.377	36	24	.880	.12988	.6996	.167	36
25	.176	.11246	.8919	.374	35	25	.908	.13017	7.6821	.163	35
26	.205	.276	.8686	.370	34	26	.937	.047	.6647	.160	34
27	.234	.305	.8455	.367	33	27	.966	.076	.6473	.156	33
28	.11263	.335	.8225	.364	32	28	.12995	.106	.6301	.152	32
29	.291	.364	.7996	.360	31	29	.13024	.136	.6129	.148	31
30	.320	.394	.7769	.99357	30	30	.053	.165	.5958	.99144	30
31	.349	.423	8.7542	.354	29	31	.081	.195	.5787	.141	29
32	.378	.452	.7317	.351	28	32	.110	.224	.5618	.137	28
33	.407	.11482	.7093	.347	27	33	.139	.13254	.5449	.133	27
34	.436	.511	.6870	.344	26	34	.168	.284	.7.5281	.129	26
35	.465	.541	.6648	.341	25	35	.197	.313	.5113	.125	25
36	.494	.570	.6427	.337	24	36	.13226	.343	.4947	.122	24
37	.11523	.600	.6208	.334	23	37	.254	.372	.4781	.118	23
38	.552	.629	.5989	.331	22	38	.283	.402	.4615	.114	22
39	.580	.659	.5772	.327	21	39	.312	.432	.4451	.110	21
40	.609	.688	.5555	.99324	20	40	.341	.461	.4287	.09106	20
41	.638	.11718	8.5340	.320	19	41	.370	.13491	.4124	.102	19
42	.667	.747	.5126	.317	18	42	.399	.521	.3962	.098	18
43	.696	.777	.4913	.314	17	43	.427	.550	.7.3800	.094	17
44	.725	.806	.4701	.310	16	44	.13456	.580	.3639	.091	16
45	.11754	.836	.4490	.307	15	45	.485	.609	.3479	.087	15
46	.783	.865	.4280	.303	14	46	.514	.639	.3319	.083	14
47	.812	.895	.4071	.300	13	47	.543	.669	.3160	.079	13
48	.840	.924	.3863	.297	12	48	.572	.698	.3002	.075	12
49	.869	.954	.3656	.293	11	49	.600	.13728	.2844	.071	11
50	.898	.11983	.3450	.99290	10	50	.629	.758	.2687	.99067	10
51	.927	.12013	8.3245	.286	9	51	.658	.787	.7.2531	.063	9
52	.956	.042	.3041	.283	8	52	.13687	.817	.2375	.059	8
53	.11985	.072	.2838	.279	7	53	.716	.846	.2220	.055	7
54	.12014	.101	.2636	.276	6	54	.744	.876	.2066	.051	6
55	.043	.131	.2434	.272	5	55	.773	.906	.1912	.047	5
56	.071	.160	.2234	.269	4	56	.802	.935	.1759	.043	4
57	.100	.190	.2035	.265	3	57	.831	.965	.1607	.039	3
58	.129	.219	.1837	.262	2	58	.860	.13995	.1455	.035	2
59	.158	.249	.1640	.258	1	59	.889	.14024	.1304	.031	1
60	.12187	.12278	8.1443	.99255	0	60	.13917	.14054	7.1154	.99027	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

83°

READ UP

82°

SOLUTION OF EQUATIONS

8°

READ DOWN

9°

/	sin	tan	cot	cos		/	sin	tan	cot	cos	
0	.13917	.14054	7.1154	.99027	60	0	.15643	.15838	6.3138	.98769	60
1	.946	.084	.1004	.023	59	1	.672	.868	.3019	.764	59
2	.13975	.113	.0855	.019	58	2	.701	.898	.2901	.760	58
3	.14004	.143	.0706	.015	57	3	.730	.928	.2783	.755	57
4	.033	.173	.0558	.011	56	4	.758	.958	.2666	.751	56
5	.061	.202	.0410	.006	55	5	.787	.15988	.2549	.746	55
6	.090	.232	.0264	.99002	54	6	.15816	.16017	.2432	.741	54
7	.119	.262	7.0117	.98998	53	7	.845	.047	.2316	.737	53
8	.148	.14291	6.9972	.994	52	8	.873	.077	.2200	.732	52
9	.177	.321	.9827	.990	51	9	.902	.107	.2085	.728	51
10	.205	.351	.9682	.986	50	10	.931	.137	.1970	.98723	50
11	.14234	.381	.9538	.982	49	11	.959	.167	.1856	.718	49
12	.263	.410	.0395	.978	48	12	.15988	.196	.1742	.714	48
13	.292	.440	.0252	.973	47	13	.16017	.226	.1628	.709	47
14	.320	.470	.0110	.969	46	14	.046	.16256	.1515	.704	46
15	.349	.499	.8969	.965	45	15	.074	.286	.1402	.700	45
16	.378	.14529	.8828	.961	44	16	.103	.316	.1290	.695	44
17	.407	.559	6.8687	.08957	43	17	.132	.346	.1178	.690	43
18	.436	.588	.8548	.953	42	18	.160	.376	.6.1066	.686	42
19	.464	.618	.8408	.948	41	19	.189	.405	.0955	.681	41
20	.14493	.648	.8269	.944	40	20	.218	.435	.0844	.98676	40
21	.522	.678	.8131	.940	39	21	.245	.465	.0734	.671	39
22	.551	.707	.7994	.936	38	22	.16275	.16495	.0624	.667	38
23	.580	.737	.7856	.931	37	23	.304	.525	.0514	.662	37
24	.608	.14757	.7720	.927	36	24	.333	.555	.0405	.657	36
25	.637	.796	6.7684	.08923	35	25	.361	.585	.0296	.652	35
26	.666	.826	.7448	.919	34	26	.390	.615	.0188	.648	34
27	.695	.856	.7313	.914	33	27	.419	.645	.6.0080	.643	33
28	.14723	.886	.7179	.910	32	28	.447	.674	.5.9972	.638	32
29	.752	.915	.7045	.906	31	29	.476	.704	.9865	.633	31
30	.781	.945	.6912	.902	30	30	.505	.16734	.0758	.98629	30
31	.810	.14975	.6779	.897	29	31	.16533	.764	.9651	.624	29
32	.838	.15005	.6646	.893	28	32	.562	.794	.9545	.619	28
33	.867	.034	.6514	.889	27	33	.591	.824	.9439	.614	27
34	.896	.064	6.6383	.08884	26	34	.620	.854	.9333	.609	26
35	.925	.094	.6252	.880	25	35	.648	.884	.9228	.604	25
36	.954	.124	.6122	.876	24	36	.677	.914	.9124	.600	24
37	.14982	.153	.5992	.871	23	37	.706	.944	.9019	.595	23
38	.15011	.183	.5863	.867	22	38	.734	.16974	.8915	.590	22
39	.040	.213	.5734	.863	21	39	.16763	.17004	.8811	.585	21
40	.069	.243	.5606	.858	20	40	.792	.033	.8708	.98580	20
41	.097	.15272	.5478	.854	19	41	.820	.063	.8605	.575	19
42	.126	.302	.5350	.849	18	42	.849	.093	.8502	.570	18
43	.155	.332	6.5223	.98845	17	43	.878	.123	.8400	.565	17
44	.184	.362	.5097	.841	16	44	.906	.153	.5.8298	.561	16
45	.15212	.391	.4971	.836	15	45	.935	.183	.8197	.556	15
46	.241	.421	.4846	.832	14	46	.964	.17213	.8095	.551	14
47	.270	.451	.4721	.827	13	47	.16992	.243	.7994	.546	13
48	.299	.481	.4596	.823	12	48	.17021	.273	.7894	.541	12
49	.327	.511	.4472	.818	11	49	.050	.303	.7794	.536	11
50	.356	.15540	.4448	.814	10	50	.078	.333	.7694	.98531	10
51	.385	.570	.4225	.809	9	51	.107	.363	.7594	.526	9
52	.15414	.600	6.4103	.98805	8	52	.136	.393	5.7495	.521	8
53	.442	.630	.3950	.800	7	53	.164	.17423	.7396	.516	7
54	.471	.660	.3859	.796	6	54	.17103	.453	.7297	.511	6
55	.500	.689	.3737	.791	5	55	.222	.483	.7199	.506	5
56	.529	.719	.3617	.787	4	56	.250	.513	.7101	.501	4
57	.557	.749	.3496	.782	3	57	.279	.543	.7004	.496	3
58	.586	.779	.3376	.778	2	58	.308	.573	.6906	.491	2
59	.615	.809	.2257	.773	1	59	.338	.603	.6809	.486	1
60	.15643	.15838	6.3138	.98769	0	60	.17365	.17633	5.6713	.98481	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

81°

READ UP

80°

TRIGONOMETRIC FUNCTIONS

255

10°

READ DOWN

11°

'	sin	tan	cot	cos		'	sin	tan	cot	cos	
cos	cot	tan	sin	'		cos	cot	tan	sin	'	
0	.17365	.17633	5.6713	.98481	60	0	.19081	.19438	5.1446	.98163	60
1	.393	.663	.6617	.476	59	1	.109	.468	.1366	.157	59
2	.422	.693	.6521	.471	58	2	.138	.498	.1286	.152	58
3	.451	.723	.6425	.466	57	3	.167	.529	.1207	.146	57
4	.479	.753	.6329	.461	56	4	.195	.559	.1128	.140	56
5	.508	.783	.6234	.455	55	5	.224	.589	.1049	.135	55
6	.537	.17813	.6140	.450	54	6	.252	.619	.0970	.129	54
7	.17565	.843	.6045	.445	53	7	.281	.649	.0892	.124	53
8	.594	.873	.5951	.440	52	8	.19309	.680	.0814	.118	52
9	.623	.903	.5857	.435	51	9	.338	.19710	5.0736	.98112	51
10	.651	.933	5.5764	.98430	50	10	.366	.740	.0658	.107	50
11	.680	.963	.5671	.425	49	11	.395	.770	.0581	.101	49
12	.708	.17993	.5578	.420	48	12	.423	.801	.0504	.096	48
13	.737	.18023	.5485	.414	47	13	.452	.831	.0427	.090	47
14	.766	.053	.5393	.409	46	14	.481	.861	.0350	.084	46
15	.17794	.083	.5301	.404	45	15	.509	.891	.0273	.079	45
16	.823	.113	.5209	.399	44	16	.19538	.921	.0197	.073	44
17	.852	.143	.5118	.394	43	17	.566	.952	.0121	.067	43
18	.880	.173	.5026	.389	42	18	.595	.19982	5.0045	.061	42
19	.909	.203	.4936	.383	41	19	.623	.20012	4.9969	.98056	41
20	.937	.233	5.4845	.98378	40	20	.652	.042	.9894	.050	40
21	.966	.18263	.4755	.373	39	21	.680	.073	.9819	.044	39
22	.17995	.293	.4665	.368	38	22	.709	.103	.9744	.039	38
23	.18023	.323	.4575	.362	37	23	.737	.133	.9669	.033	37
24	.052	.353	.4486	.357	36	24	.19766	.164	.9594	.027	36
25	.081	.384	.4397	.352	35	25	.794	.194	.9520	.021	35
26	.109	.414	.4308	.347	34	26	.823	.20224	.9446	.016	34
27	.138	.444	.4219	.341	33	27	.851	.254	4.9372	.010	33
28	.166	.474	.4131	.336	32	28	.880	.285	.9298	.98004	32
29	.195	.18504	.4043	.331	31	29	.908	.315	.9225	.97998	31
30	.224	.534	5.3955	.98325	30	30	.937	.345	.9152	.992	30
31	.252	.564	.3868	.320	29	31	.965	.376	.9078	.987	29
32	.18281	.594	.3781	.315	28	32	.19994	.406	.9006	.981	28
33	.309	.624	.3694	.310	27	33	.20022	.436	.8933	.975	27
34	.338	.654	.3607	.304	26	34	.051	.20466	.8860	.969	26
35	.367	.684	.3521	.299	25	35	.079	.497	4.8788	.963	25
36	.395	.714	.3435	.294	24	36	.108	.527	.8716	.958	24
37	.424	.18745	.3349	.288	23	37	.136	.557	.8644	.97952	23
38	.452	.775	.3263	.283	22	38	.165	.588	.8573	.946	22
39	.481	.805	.3178	.277	21	39	.193	.618	.8501	.940	21
40	.18509	.835	5.3093	.98272	20	40	.222	.648	.8430	.934	20
41	.538	.865	.3008	.267	19	41	.250	.679	.8359	.928	19
42	.567	.895	.2924	.261	18	42	.20279	.709	.8288	.922	18
43	.595	.925	.2839	.256	17	43	.307	.20739	.8218	.916	17
44	.624	.955	.2755	.250	16	44	.336	.770	4.8147	.97910	16
45	.652	.18986	.2672	.245	15	45	.364	.800	.8077	.905	15
46	.681	.19016	.2588	.240	14	46	.393	.830	.8007	.899	14
47	.710	.046	.2505	.234	13	47	.421	.861	.7937	.893	13
48	.738	.076	.2422	.229	12	48	.450	.891	.7867	.887	12
49	.18767	.106	.2339	.223	11	49	.478	.921	.7798	.881	11
50	.765	.136	5.2257	.98218	10	50	.507	.952	.7729	.875	10
51	.824	.166	.2174	.212	9	51	.20535	.20982	.7659	.869	9
52	.852	.197	.2092	.207	8	52	.563	.21013	4.7591	.97863	8
53	.881	.19227	.2011	.201	7	53	.592	.043	.7522	.857	7
54	.910	.257	.1929	.196	6	54	.620	.073	.7453	.851	6
55	.938	.287	.1848	.190	5	55	.649	.104	.7385	.845	5
56	.967	.317	.1767	.185	4	56	.677	.134	.7317	.839	4
57	.18995	.347	.1686	.179	3	57	.706	.164	.7249	.833	3
58	.19024	.378	.1606	.174	2	58	.734	.195	.7181	.827	2
59	.052	.408	.1526	.168	1	59	.763	.225	.7114	.821	1
60	.19051	.19438	5.1446	.98163	0	60	.20791	.21256	4.7046	.97815	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'

79°

READ UP

78°

SOLUTION OF EQUATIONS

12°

READ DOWN

13°

	sin	tan	cot	cos			sin	tan	cot	cos		
	cos	cot	tan	sin			cos	cot	tan	sin		
0	.20791	.21256	4.7046	.97815	60	0	.22495	.23087	4.3315	.97437	60	
1	.820	.286	4.6979	.809	59	1	.523	.117	.257	.430	59	
2	.848	.316	912	.803	58	2	.552	.148	.200	.424	58	
3	.877	.347	815	.797	57	3	.580	.179	.143	.417	57	
4	.905	.377	779	.791	56	4	.608	.209	.086	.411	56	
5	.933	.408	712	.784	55	5	.637	.240	4.3029	.404	55	
6	.962	.438	646	.778	54	6	.665	.271	4.2972	.398	54	
7	.20990	.469	580	.772	53	7	.693	.23301	.916	.391	53	
8	.21019	.21499	4.6514	.766	52	8	.22722	.332	.859	.351	52	
9	.047	.529	448	.760	51	9	.750	.363	.803	.378	51	
10	.076	.560	382	.97754	50	10	.778	.393	.747	.97371	50	
11	.101	.590	317	.748	49	11	.807	.424	.691	.365	49	
12	.132	.621	252	.742	48	12	.835	.455	.635	.358	48	
13	.161	.651	187	.735	47	13	.863	.485	.580	.351	47	
14	.189	.682	122	.729	46	14	.892	.516	4.2524	.345	46	
15	.218	.712	4.6057	.723	45	15	.920	.23547	.468	.338	45	
16	.21246	.21743	4.5993	.717	44	16	.948	.578	.413	.331	44	
17	.275	.773	928	.711	43	17	.22977	.608	.358	.325	43	
18	.303	.804	864	.705	42	18	.23005	.639	.303	.318	42	
19	.331	.834	800	.698	41	19	.033	.670	.248	.311	41	
20	.360	.864	736	.97692	40	20	.062	.700	.193	.97304	40	
21	.388	.895	673	.686	39	21	.090	.731	.139	.298	39	
22	.417	.925	609	.680	38	22	.118	.23762	.084	.291	38	
23	.445	.956	4.5546	.673	37	23	.146	.793	4.2030	.284	37	
24	.474	.21986	483	.667	36	24	.175	.823	4.1976	.278	36	
25	.21502	.22017	420	.661	35	25	.203	.851	.922	.271	35	
26	.530	.047	.357	.655	34	26	.231	.885	.868	.264	34	
27	.559	.078	.294	.618	33	27	.23260	.916	.814	.257	33	
28	.587	.108	.232	.642	32	28	.238	.946	.760	.251	32	
29	.616	.139	.160	.636	31	29	.29	.316	.23977	.708	.244	31
30	.645	.169	.107	.97630	30	30	.315	.21008	.653	.97237	30	
31	.672	.200	4.5015	.623	29	31	.373	.039	.600	.230	29	
32	.701	.231	4.4983	.617	28	32	.401	.069	.547	.223	28	
33	.729	.22261	.922	.611	27	33	.420	.100	4.1493	.217	27	
34	.21758	.292	.860	.604	26	34	.458	.131	.441	.210	26	
35	.786	.322	.709	.698	25	35	.23486	.162	.388	.203	25	
36	.814	.353	.737	.592	24	36	.514	.193	.335	.196	24	
37	.813	.383	.676	.585	23	37	.542	.223	.282	.189	23	
38	.871	.414	.615	.579	22	38	.571	.24251	.230	.182	22	
39	.893	.444	.555	.573	21	39	.599	.285	.178	.176	21	
40	.928	.475	4.4494	.07566	20	40	.627	.316	.126	.97169	20	
41	.056	.22505	434	.560	19	41	.656	.347	.074	.162	19	
42	.21985	.536	.373	.553	18	42	.684	.377	4.1022	.155	18	
43	.22013	.567	.313	.547	17	43	.712	.408	4.0970	.148	17	
44	.041	.597	.253	.541	16	44	.23740	.439	.918	.141	16	
45	.070	.628	.194	.531	15	45	.769	.24470	.867	.134	15	
46	.098	.658	.134	.528	14	46	.797	.501	.815	.127	14	
47	.126	.689	.075	.521	13	47	.825	.532	.764	.120	13	
48	.155	.719	4.4015	.515	12	48	.853	.562	.713	.113	12	
49	.183	.22750	4.3956	.508	11	49	.882	.593	.662	.106	11	
50	.212	.781	.807	.07502	10	50	.910	.624	.611	.97100	10	
51	.210	.811	.838	.496	9	51	.938	.655	.560	.093	9	
52	.22268	.812	.779	.489	8	52	.966	.24686	4.0509	.086	8	
53	.297	.872	.721	.483	7	53	.24995	.717	.450	.079	7	
54	.325	.903	.662	.476	6	54	.24023	.747	.408	.072	6	
55	.353	.034	.604	.470	5	55	.051	.778	.258	.065	5	
56	.382	.964	.516	.463	4	56	.079	.809	.308	.053	4	
57	.410	.22905	.488	.457	3	57	.108	.840	.257	.051	3	
58	.438	.23026	.430	.450	2	58	.136	.871	.207	.044	2	
59	.467	.056	.372	.444	1	59	.161	.902	.158	.037	1	
60	.23495	.23087	4.3315	.97437	0	60	.21192	.21933	4.0108	.97030	0	
	cos	cot	tan	sin	/		cos	cot	tan	sin	/	

77°

READ UP

78°

TRIGONOMETRIC FUNCTIONS

257

14°

READ DOWN

15°

'	sin	tan	cot	cos	
0	.24192	.24933	4.0108	.97030	60
1	.220	.964	.058	.023	59
2	.249	.24995	4.0009	.015	58
3	.277	.25026	3.9959	.008	57
4	.305	.056	.910	.97001	56
5	.333	.087	.861	.96994	55
6	.362	.118	.812	.987	54
7	.390	.149	.763	.980	53
8	.418	.180	.714	.973	52
9	.446	.211	.665	.966	51
10	.474	.242	.617	.959	50
11	.503	.25273	.568	.952	49
12	.531	.304	3.9520	.945	48
13	.559	.335	.471	.96937	47
14	.587	.366	.423	.930	46
15	.615	.397	.375	.923	45
16	.644	.428	.327	.916	44
17	.672	.459	.279	.909	43
18	.700	.25490	.232	.902	42
19	.24728	.521	.184	.894	41
20	.756	.552	.136	.887	40
21	.784	.583	.089	.96880	39
22	.813	.614	3.9042	.873	38
23	.841	.645	3.8995	.866	37
24	.869	.676	.947	.858	36
25	.897	.707	.900	.851	35
26	.925	.25738	.854	.844	34
27	.954	.769	.807	.837	33
28	.24982	.800	.760	.829	32
29	.25010	.831	.714	.96822	31
30	.038	.862	3.8667	.815	30
31	.066	.893	.621	.807	29
32	.094	.924	.575	.800	28
33	.122	.955	.528	.793	27
34	.151	.25986	.482	.786	26
35	.179	.26017	.436	.778	25
36	.207	.048	.391	.771	24
37	.25235	.079	3.8345	.96764	23
38	.263	.110	.299	.756	22
39	.291	.141	.254	.749	21
40	.320	.172	.208	.742	20
41	.348	.203	.163	.734	19
42	.376	.235	.118	.727	18
43	.404	.26266	.073	.719	17
44	.25432	.297	3.8028	.712	16
45	.400	.328	3.7983	.96705	15
46	.488	.359	.938	.697	14
47	.516	.390	.893	.690	13
48	.545	.421	.848	.682	12
49	.573	.452	.804	.675	11
50	.601	.483	.760	.667	10
51	.629	.515	.715	.660	9
52	.25657	.26546	3.7671	.653	8
53	.655	.577	.627	.96645	7
54	.713	.608	.583	.636	6
55	.741	.639	.539	.630	5
56	.769	.670	.495	.623	4
57	.798	.701	.451	.615	3
58	.826	.733	.408	.608	2
59	.854	.764	.364	.600	1
60	.25582	.26795	3.7321	.96593	0
	cos	cot	tan	sin	'

75°

READ UP

74°

'	sin	tan	cot	cos	
0	.25882	.26795	3.7321	.96593	60
1	.910	.826	.277	.585	59
2	.938	.857	.234	.578	58
3	.966	.888	.191	.570	57
4	.25994	.920	.148	.562	56
5	.26022	.951	.105	.555	55
6	.050	.26982	.062	.547	54
7	.079	.27013	3.7019	.540	53
8	.107	.044	3.6976	.532	52
9	.135	.076	.933	.524	51
10	.163	.107	.891	.96517	50
11	.191	.138	.848	.509	49
12	.219	.169	.806	.502	48
13	.26247	.201	.764	.494	47
14	.275	.232	.722	.486	46
15	.303	.27263	.680	.479	45
16	.331	.294	3.6638	.471	44
17	.359	.326	.596	.463	43
18	.387	.357	.554	.456	42
19	.415	.388	.512	.448	41
20	.443	.419	.470	.96440	40
21	.26471	.451	.429	.433	39
22	.500	.27482	.387	.425	38
23	.528	.513	.346	.417	37
24	.556	.545	3.6305	.410	36
25	.584	.576	.264	.402	35
26	.612	.607	.222	.394	34
27	.640	.638	.181	.386	33
28	.668	.670	.140	.379	32
29	.696	.701	.100	.371	31
30	.26724	.27732	.059	.96363	30
31	.752	.764	3.6018	.355	29
32	.780	.795	3.5978	.347	28
33	.808	.826	.937	.340	27
34	.836	.858	.897	.332	26
35	.864	.889	.856	.324	25
36	.892	.921	.816	.316	24
37	.920	.952	.776	.308	23
38	.948	.27983	.736	.301	22
39	.26976	.28015	.696	.293	21
40	.27004	.046	3.5656	.96285	20
41	.032	.077	.616	.277	19
42	.060	.109	.576	.269	18
43	.088	.140	.536	.261	17
44	.116	.172	.497	.253	16
45	.144	.203	.457	.246	15
46	.172	.28234	.418	.238	14
47	.200	.266	.379	.230	13
48	.228	.297	3.5339	.222	12
49	.256	.329	.300	.214	11
50	.27284	.360	.261	.96206	10
51	.312	.391	.222	.198	9
52	.340	.423	.183	.190	8
53	.368	.28454	.144	.182	7
54	.396	.486	.105	.174	6
55	.424	.517	.067	.166	5
56	.452	.549	3.5028	.158	4
57	.480	.580	3.4989	.150	3
58	.508	.612	.951	.142	2
59	.536	.643	.912	.134	1
60	.27564	.28075	3.4874	.96126	0
	cos	cot	tan	sin	'

SOLUTION OF EQUATIONS

16°

READ DOWN

17°

	sin	tan	cot	cos		sin	tan	cot	cos		
0	.27564	.28675	3.4574	.96126	60	0	.39237	.30573	3.2709	.95630	60
1	.592	.706	.836	.118	59	1	.265	.605	.675	.622	59
2	.620	.738	.798	.110	58	2	.293	.637	.641	.613	58
3	.648	.769	.760	.102	57	3	.321	.669	.607	.605	57
4	.676	.801	.722	.094	56	4	.348	.700	.573	.596	56
5	.704	.832	.684	.086	55	5	.376	.732	.539	.588	55
6	.731	.864	.646	.078	54	6	.404	.764	.506	.579	54
7	.27759	.895	.608	.96070	53	7	.432	.30796	3.2172	.571	53
8	.787	.927	3.4570	.062	52	8	.460	.828	.438	.562	52
9	.815	.958	.533	.054	51	9	.29487	.860	.405	.554	51
10	.843	.28990	495	.046	50	10	.515	.891	.371	.95545	50
11	.871	.29021	458	.037	49	11	.543	.923	.338	.536	49
12	.899	.053	420	.029	48	12	.571	.955	.305	.525	48
13	.927	.084	383	.021	47	13	.599	.30987	.272	.519	47
14	.955	.116	346	.013	46	14	.626	.31019	3.2238	.511	46
15	.27983	.147	3.4308	.96005	45	15	.654	.051	.205	.502	45
16	.28011	.179	.271	.95997	44	16	.682	.083	.172	.493	44
17	.039	.210	.234	.989	43	17	.710	.115	.139	.483	43
18	.067	.29242	.197	.981	42	18	.29737	.147	.106	.476	42
19	.095	.274	.160	.972	41	19	.765	.178	.073	.467	41
20	.123	.305	.124	.964	40	20	.793	.210	.041	.95459	40
21	.150	.337	.087	.956	39	21	.821	.242	3.2008	.450	39
22	.178	.368	.050	.948	38	22	.849	.31274	3.1975	.441	38
23	.206	.400	3.4014	.940	37	23	.876	.206	.943	.433	37
24	.28234	.432	3.3977	.931	36	24	.904	.338	.910	.424	36
25	.262	.29463	.941	.95923	35	25	.932	.370	.878	.415	35
26	.290	.495	.904	.915	34	26	.960	.402	.545	.407	34
27	.318	.526	.868	.907	33	27	.29987	.434	.813	.398	33
28	.346	.558	.832	.898	32	28	.30015	.466	.780	.389	32
29	.374	.590	.796	.890	31	29	.043	.31408	3.1748	.380	31
30	.402	.621	.759	.882	30	30	.071	.630	.716	.95372	30
31	.429	.653	.723	.874	29	31	.098	.562	.684	.363	29
32	.457	.685	.687	.865	28	32	.126	.594	.652	.351	28
33	.28485	.29716	3.3652	.95857	27	33	.154	.626	.620	.345	27
34	.513	.748	.616	.849	26	34	.182	.658	.588	.337	26
35	.541	.780	.580	.841	25	35	.209	.690	.556	.325	25
36	.569	.811	.544	.832	24	36	.237	.722	.524	.319	24
37	.597	.843	.509	.824	23	37	.30265	.31754	3.1492	.310	23
38	.625	.875	.473	.816	22	38	.292	.786	.460	.301	22
39	.652	.906	.438	.807	21	39	.320	.818	.429	.293	21
40	.680	.935	.402	.799	20	40	.348	.850	.397	.95284	20
41	.708	.29970	.367	.791	19	41	.376	.882	.366	.275	19
42	.28736	.30001	3.3332	.95782	18	42	.403	.914	.334	.260	18
43	.784	.033	.297	.774	17	43	.431	.946	.303	.257	17
44	.792	.065	.261	.766	16	44	.459	.31978	3.1271	.245	16
45	.820	.097	.226	.757	15	45	.30486	.32010	.240	.240	15
46	.847	.125	.191	.749	14	46	.514	.642	.209	.231	14
47	.875	.160	.156	.740	13	47	.542	.074	.178	.222	13
48	.903	.192	.122	.732	12	48	.570	.105	.146	.213	12
49	.931	.224	.087	.724	11	49	.597	.139	.115	.204	11
50	.959	.255	.052	.715	10	50	.625	.171	.084	.95195	10
51	.28987	.30287	3.3017	.05707	9	51	.653	.203	.053	.186	9
52	.29015	.319	3.2983	.695	8	52	.680	.235	3.1022	.177	8
53	.042	.351	.948	.690	7	53	.30708	.32267	3.0991	.165	7
54	.070	.382	.914	.681	6	54	.736	.299	.961	.159	6
55	.098	.414	.879	.673	5	55	.763	.331	.930	.150	5
56	.126	.446	.845	.664	4	56	.791	.363	.899	.142	4
57	.154	.478	.811	.656	3	57	.819	.395	.868	.133	3
58	.182	.509	.777	.647	2	58	.846	.425	.835	.124	2
59	.209	.541	.743	.639	1	59	.874	.460	.807	.115	1
60	.29237	.30573	3.2709	.95630	0	60	.30902	.32492	3.0777	.95106	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

73°

READ UP

72°

18° READ DOWN 19°

'	sin	tan	cot	cos		'	sin	tan	cot	cos	
0	.30902	.32492	3.0777	.95106	60	0	.32557	.34433	2.9042	.94552	60
1	.929	.524	.746	.097	.59	1	.584	.465	2.9015	.542	.59
2	.957	.556	.716	.088	.58	2	.612	.498	2.8987	.533	.58
3	.30985	.588	.686	.079	.57	3	.639	.530	.960	.523	.57
4	.31012	.621	.655	.070	.56	4	.667	.563	.933	.514	.56
5	.040	.653	.625	.061	.55	5	.694	.596	.905	.504	.55
6	.068	.685	.595	.95052	.54	6	.722	.628	.878	.495	.54
7	.095	.717	.565	.043	.53	7	.749	.661	.851	.485	.53
8	.123	.32749	.535	.033	.52	8	.32777	.693	.824	.476	.52
9	.151	.782	3.0505	.024	.51	9	.804	.34726	.797	.466	.51
10	.178	.814	.475	.015	.50	10	.832	.758	.770	.94457	50
11	.206	.846	.445	.95006	.49	11	.859	.791	2.8743	.447	.49
12	.233	.878	.415	.94997	.48	12	.887	.824	.716	.438	.48
13	.261	.911	.385	.988	.47	13	.914	.856	.689	.428	.47
14	.31289	.943	.356	.979	.46	14	.942	.889	.662	.418	.46
15	.316	.32975	.326	.970	.45	15	.969	.922	.636	.409	.45
16	.344	.33007	.296	.961	.44	16	.32997	.954	.609	.399	.44
17	.372	.040	3.0267	.952	.43	17	.33024	.34987	.582	.390	.43
18	.399	.072	.237	.943	.42	18	.051	.35020	.556	.380	.42
19	.427	.104	.208	.933	.41	19	.079	.052	.529	.370	.41
20	.454	.136	.178	.94924	.40	20	.106	.085	2.8502	.94361	40
21	.482	.169	.149	.915	.39	21	.134	.118	.476	.351	.39
22	.31510	.201	.120	.906	.38	22	.161	.150	.449	.342	.38
23	.537	.233	.090	.897	.37	23	.189	.183	.423	.332	.37
24	.565	.33266	.061	.888	.36	24	.33216	.216	.397	.322	.36
25	.593	.298	.032	.878	.35	25	.244	.35248	.370	.313	.35
26	.620	.330	3.0003	.869	.34	26	.271	.281	.344	.303	.34
27	.648	.363	2.9974	.860	.33	27	.298	.314	.318	.293	.33
28	.675	.395	.945	.94851	.32	28	.326	.346	.291	.284	.32
29	.703	.427	.916	.842	.31	29	.353	.379	2.8265	.274	.31
30	.730	.460	.887	.832	.30	30	.381	.412	.239	.94264	30
31	.31758	.33492	.858	.823	.29	31	.33408	.35445	.213	.254	.29
32	.786	.524	.829	.814	.28	32	.436	.477	.187	.245	.28
33	.813	.557	.800	.805	.27	33	.463	.510	.161	.235	.27
34	.841	.589	.772	.795	.26	34	.490	.543	.135	.225	.26
35	.868	.621	2.9743	.786	.25	35	.518	.576	.109	.215	.25
36	.896	.654	.714	.94777	.24	36	.545	.608	.083	.206	.24
37	.923	.686	.686	.768	.23	37	.573	.641	.057	.196	.23
38	.951	.33718	.657	.758	.22	38	.33600	.674	.032	.186	.22
39	.31979	.751	.629	.749	.21	39	.627	.35707	2.8006	.176	.21
40	.32006	.783	.600	.740	.20	40	.655	.740	2.7980	.94167	20
41	.034	.816	.572	.730	.19	41	.682	.772	.955	.157	.19
42	.061	.848	.544	.721	.18	42	.710	.805	.929	.147	.18
43	.089	.881	.515	.712	.17	43	.737	.838	.903	.137	.17
44	.116	.913	2.9487	.94702	.16	44	.764	.871	.878	.127	.16
45	.144	.945	.459	.693	.15	45	.33792	.904	.852	.118	.15
46	.171	.33978	.431	.684	.14	46	.819	.937	2.7827	.108	.14
47	.199	.34010	.403	.674	.13	47	.846	.35969	.801	.098	.13
48	.227	.043	.375	.665	.12	48	.874	.36002	.776	.94088	.12
49	.254	.075	.347	.656	.11	49	.901	.035	.751	.078	.11
50	.32282	.108	.319	.646	.10	50	.929	.068	.725	.068	.10
51	.309	.140	2.9291	.637	.9	51	.956	.101	.700	.058	.9
52	.337	.173	.263	.94627	.8	52	.33983	.134	.675	.049	.8
53	.364	.205	.235	.618	.7	53	.34011	.167	2.7650	.039	.7
54	.392	.34338	.208	.609	.6	54	.038	.36199	.625	.029	.6
55	.419	.270	.180	.599	.5	55	.065	.232	.600	.019	.5
56	.447	.303	.152	.590	.4	56	.093	.265	.575	.94009	.4
57	.474	.335	.125	.580	.3	57	.120	.298	.550	.93909	.3
58	.502	.368	.097	.571	.2	58	.147	.331	.525	.989	.2
59	.529	.400	.070	.561	.1	59	.175	.364	.500	.979	.1
60	.32557	.34433	2.9042	.94552	0	60	.34202	.36397	2.7475	.93969	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'

71°

READ UP

70°

SOLUTION OF EQUATIONS

20°

READ DOWN

21°

	sin	tan	cot	cos			sin	tan	cot	cos		
0	.34202	.36307	2.7475	.93969	60		0	.35837	.38386	2.6051	.93356	60
1	229	430	450	959	59		1	864	420	028	348	59
2	257	463	425	949	58		2	891	453	2.6006	337	58
3	234	496	400	930	57		3	918	487	2.5983	327	57
4	311	529	376	929	56		4	915	520	961	310	56
5	339	562	351	919	55		5	.35073	553	938	306	55
6	366	595	326	909	54		6	.36000	587	916	295	54
7	.34393	628	302	899	53		7	937	620	893	285	53
8	421	661	277	880	52		8	954	654	871	.93274	52
9	448	.36604	2.7253	879	51		9	981	.38687	818	264	51
10	475	727	228	.93869	50		10	108	721	826	253	50
11	503	760	201	.859	49		11	135	754	2.5804	243	49
12	530	793	179	.849	48		12	162	787	782	232	48
13	557	826	155	.839	47		13	190	821	750	222	47
14	.34584	859	130	.829	46		14	217	854	737	211	46
15	612	802	106	.819	45		15	.36214	888	715	201	45
16	639	925	052	.809	44		16	271	921	.93190	44	
17	666	958	058	.799	43		17	298	955	671	180	43
18	694	.36991	034	.789	42		18	325	.38988	2.5649	169	42
19	721	.37024	2.7009	.779	41		19	352	.39022	627	150	41
20	748	057	2.6985	.93769	40		20	379	055	605	145	40
21	.34775	090	961	759	39		21	406	089	553	137	39
22	803	123	937	748	38		22	434	122	561	127	38
23	830	157	913	735	37		23	461	156	539	116	37
24	857	190	889	725	36		24	.36488	190	517	106	36
25	884	223	865	718	35		25	515	223	2.5195	.93095	35
26	912	.37256	841	708	34		26	542	.39257	473	084	34
27	939	289	818	698	33		27	569	290	452	074	33
28	966	322	.2.6794	688	32		28	596	324	430	063	32
29	.34993	355	770	677	31		29	623	357	408	052	31
30	.35021	388	746	.93667	30		30	650	391	386	012	30
31	018	422	723	657	29		31	677	425	365	031	29
32	078	455	699	647	28		32	704	458	.2.5313	020	28
33	102	.37185	675	637	27		33	.36731	.39492	322	.93010	27
34	130	521	652	626	26		34	758	526	300	.92999	26
35	157	551	628	616	25		35	785	559	279	998	25
36	184	588	.2.6605	606	24		36	812	593	257	978	24
37	.35211	621	581	596	23		37	839	626	236	967	23
38	239	654	558	585	22		38	867	660	214	956	22
39	266	687	534	575	21		39	894	694	193	945	21
40	293	.37720	511	.93565	20		40	921	.39727	2.5172	035	20
41	320	754	488	555	19		41	948	761	150	021	19
42	347	787	464	544	18		42	.36075	795	129	913	18
43	375	820	441	534	17		43	.37002	829	108	.02902	17
44	.35102	853	.2.6418	524	16		44	929	862	086	892	16
45	429	887	395	514	15		45	956	896	065	881	15
46	456	920	371	503	14		46	983	930	014	870	14
47	484	953	348	493	13		47	110	963	023	850	13
48	511	.37980	325	483	12		48	137	.39997	2.5002	819	12
49	538	.38020	302	472	11		49	161	.40031	2.4981	838	11
50	565	033	279	.93162	10		50	191	045	960	827	10
51	.35592	056	256	452	9		51	218	098	939	816	9
52	619	120	.2.6233	441	8		52	.37215	132	918	.92505	8
53	647	153	210	431	7		53	272	166	897	791	7
54	674	186	187	420	6		54	299	200	876	784	6
55	701	220	165	410	5		55	326	231	2.4855	773	5
56	728	253	142	400	4		56	353	267	834	702	4
57	755	286	110	399	3		57	380	301	813	751	3
58	782	320	096	379	2		58	407	335	792	710	2
59	810	353	074	369	1		59	431	369	772	729	1
60	.35837	.38396	.2.6051	.93356	0		60	.37461	.40103	2.4751	.92718	0
	cos	cot	tan	sin	/			cos	cot	tan	sin	/

69°

READ UP

68°

TRIGONOMETRIC FUNCTIONS

261

READ DOWN					
cot	cos	'	sin	tan	cot
.03	2.4751	.92718	60	.39073	.42447
.36	730	707	59	1	100
.70	709	697	58	2	127
.04	689	686	57	3	153
.38	668	675	56	4	180
.72	648	664	55	5	207
.06	627	653	54	6	234
.40	606	642	53	7	260
.74	586	631	52	8	287
.07	2.4566	620	51	9	.39314
.41	545	.92609	50	10	341
.75	525	598	49	11	367
.09	504	587	48	12	394
.43	484	576	47	13	421
.77	464	565	46	14	448
.11	443	554	45	15	474
.45	423	543	44	16	501
.79	403	532	43	17	.42998
.13	2.4383	521	42	18	.43032
.47	362	510	41	19	.39555
.81	342	.92499	40	20	608
.15	322	488	39	21	635
.49	302	477	38	22	661
.83	282	466	37	23	688
.17	262	455	36	24	715
.51	242	444	35	25	741
.85	222	432	34	26	.39768
.19	2.4202	421	33	27	.43343
.53	182	410	32	28	795
.87	162	399	31	29	378
.21	142	.92388	30	30	822
.55	122	377	29	31	848
.90	102	366	28	32	902
.24	083	355	27	33	928
.58	063	343	26	34	.39982
.92	043	332	25	35	.40008
.26	023	321	24	36	.43654
.60	2.4004	310	23	37	035
.94	2.3984	299	22	38	689
.28	964	287	21	39	724
.63	945	.92276	20	40	2.2871
.97	925	265	19	41	.91625
.31	906	254	18	42	2.2998
.65	886	243	17	43	709
.99	867	231	16	44	962
.33	2.3847	220	15	45	980
.68	828	209	14	46	992
.02	808	198	13	47	1000
.36	789	186	12	48	1012
.70	770	175	11	49	1024
.05	750	.92164	10	50	1036
.39	731	152	9	51	1050
.73	2.3712	141	8	52	1064
.07	693	130	7	53	1078
.42	673	119	6	54	1092
.76	654	107	5	55	1106
.10	635	096	4	56	1120
.45	616	085	3	57	1134
.79	597	073	2	58	1148
.13	578	062	1	59	1162
.47	2.3559	.92050	0	60	1176
	tan	sin	'	cos	cot
				tan	sin

67°

READ UP

66°

SOLUTION OF EQUATIONS

24°

READ DOWN

25°

	sin	tan	cot	cos		sin	tan	cot	cos		
0	.40674	.44523	2.2460	.91355	60	0	.42262	.46631	2.1445	.90631	60
1	.700	.555	443	.343	59	1	.288	.666	.429	.618	59
2	.727	.593	425	.331	58	2	.315	.702	.413	.606	58
3	.753	.627	408	.319	57	3	.341	.737	.396	.594	57
4	.780	.662	390	.307	56	4	.367	.772	.380	.582	56
5	.806	.697	373	.295	55	5	.394	.808	.364	.569	55
6	.40833	.732	355	.283	54	6	.420	.843	.318	.557	54
7	.860	.44767	338	.91272	53	7	.42446	.879	.332	.545	53
8	.886	.802	320	.260	52	8	.473	.914	.315	.532	52
9	.913	.837	2.2303	.218	51	9	.499	.950	2.1299	.520	51
10	.939	.872	286	.236	50	10	.525	.46985	.283	.90507	50
11	.966	.907	268	.224	49	11	.552	.47021	.267	.495	49
12	.40992	.912	251	.212	48	12	.578	.036	.251	.483	48
13	.41019	.44977	234	.200	47	13	.604	.092	.235	.470	47
14	.045	.45012	216	.188	46	14	.42631	.128	.219	.458	46
15	.072	.047	190	.91176	45	15	.657	.163	.203	.446	45
16	.098	.082	182	.104	44	16	.683	.199	.187	.433	44
17	.125	.117	2.2165	.152	43	17	.709	.234	.171	.421	43
18	.151	.152	148	.140	42	18	.736	.270	2.1155	.408	42
19	.178	.187	130	.128	41	19	.762	.305	.139	.396	41
20	.204	.222	113	.116	40	20	.788	.47341	.123	.90383	40
21	.231	.45257	096	.104	39	21	.42815	.377	.107	.371	39
22	.41257	.292	.079	.91092	38	22	.841	.412	.092	.358	38
23	.284	.327	.062	.050	37	23	.867	.446	.076	.346	37
24	.310	.362	.045	.068	36	24	.894	.483	.060	.334	36
25	.337	.397	.026	.056	35	25	.920	.519	.044	.321	35
26	.363	.432	2.2011	.044	34	26	.946	.555	.026	.309	34
27	.390	.467	2.1994	.032	33	27	.972	.590	2.1013	.296	33
28	.416	.45502	977	.020	32	28	.42993	.626	2.0997	.284	32
29	.443	.538	.960	.91008	31	29	.43025	.47662	.981	.271	31
30	.469	.573	.943	.90996	30	30	.051	.698	.965	.90250	30
31	.41496	.605	.926	.984	29	31	.077	.733	.950	.246	29
32	.522	.643	.909	.972	28	32	.104	.769	.934	.233	28
33	.549	.678	.892	.900	27	33	.130	.805	.918	.221	27
34	.575	.713	.876	.948	26	34	.156	.840	.903	.208	26
35	.602	.45748	2.1850	.036	25	35	.182	.876	.887	.196	25
36	.628	.784	.812	.924	24	36	.209	.912	2.0872	.183	24
37	.655	.819	.825	.90311	23	37	.43235	.948	.856	.171	23
38	.681	.851	.805	.899	22	38	.261	.47984	.840	.158	22
39	.707	.889	.792	.887	21	39	.287	.48019	.825	.146	21
40	.41734	.924	.775	.875	20	40	.313	.055	.809	.90133	20
41	.760	.960	.755	.863	19	41	.340	.091	.794	.120	19
42	.787	.45995	.742	.851	18	42	.366	.127	.776	.108	18
43	.813	.46030	2.1725	.839	17	43	.392	.163	.763	.095	17
44	.840	.005	.708	.826	16	44	.43418	.193	2.0748	.082	16
45	.866	.101	.092	.814	15	45	.415	.234	.732	.070	15
46	.892	.136	.075	.00502	14	46	.471	.45270	.717	.057	14
47	.919	.171	.659	.790	13	47	.497	.306	.701	.045	13
48	.945	.206	.642	.778	12	48	.523	.312	.686	.032	12
49	.972	.212	.625	.766	11	49	.549	.378	.671	.019	11
50	.41995	.277	.609	.753	10	50	.575	.414	.055	.90007	10
51	.42024	.312	.592	.741	9	51	.602	.450	.640	.89994	9
52	.051	.46348	2.1576	.729	8	52	.43028	.486	2.0625	.931	8
53	.077	.353	.560	.00717	7	53	.054	.48521	.609	.968	7
54	.104	.418	.513	.701	6	54	.680	.557	.591	.956	6
55	.130	.451	.527	.692	5	55	.706	.593	.579	.913	5
56	.156	.480	.510	.680	4	56	.733	.629	.584	.930	4
57	.183	.525	.491	.668	3	57	.759	.665	.549	.918	3
58	.209	.560	.478	.655	2	58	.785	.701	.533	.905	2
59	.235	.595	.461	.643	1	59	.811	.737	.518	.892	1
60	.42262	.46631	2.1443	.90631	0	60	.43837	.48773	2.0503	.89370	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

65°

READ UP

64°

TRIGONOMETRIC FUNCTIONS

263

26°

READ DOWN

27°

'	sin	tan	cot	cos		'	sin	tan	cot	cos	
cos	cot	tan	sin	'		cos	cot	tan	sin	'	
0	.43837	.48773	2.0503	.89879	60	0	.45399	.50953	1.9626	.89101	60
1	.863	.809	488	.867	59	1	.425	.50989	612	.087	59
2	.889	.845	473	.854	58	2	.451	.51026	598	.074	58
3	.916	.881	458	.841	57	3	.477	.063	584	.061	57
4	.942	.917	443	.828	56	4	.503	.099	570	.048	56
5	.968	.953	428	.816	55	5	.529	.136	556	.035	55
6	.43994	.48989	413	.803	54	6	.554	.173	542	.021	54
7	.44020	.49026	398	.790	53	7	.45580	.209	528	.89008	53
8	.046	.062	2.0383	.777	52	8	.606	.246	1.9514	.88995	52
9	.072	.098	368	.764	51	9	.632	.283	500	.981	51
10	.098	.134	353	.89752	50	10	.658	.319	486	.968	50
11	.124	.170	338	.730	49	11	.684	.51356	472	.955	49
12	.151	.206	323	.726	48	12	.710	.393	458	.942	48
13	.177	.242	308	.713	47	13	.736	.430	444	.928	47
14	.203	.278	293	.700	46	14	.762	.467	430	.915	46
15	.44220	.49315	2.0278	.687	45	15	.45787	.503	1.9416	.902	45
16	.255	.351	263	.674	44	16	.813	.540	402	.88888	44
17	.281	.387	248	.662	43	17	.839	.577	388	.875	43
18	.307	.423	233	.649	42	18	.865	.614	375	.862	42
19	.333	.459	219	.636	41	19	.891	.651	361	.848	41
20	.359	.495	204	.89623	40	20	.917	.51688	347	.835	40
21	.385	.532	189	.610	39	21	.942	.724	333	.822	39
22	.44411	.568	174	.597	38	22	.968	.761	1.9319	.808	38
23	.437	.604	160	.584	37	23	.45994	.798	306	.795	37
24	.464	.49640	2.0145	.571	36	24	.46020	.835	292	.88782	36
25	.490	.677	130	.558	35	25	.046	.872	278	.768	35
26	.516	.713	115	.545	34	26	.072	.909	265	.755	34
27	.542	.749	101	.532	33	27	.097	.946	251	.741	33
28	.568	.786	086	.519	32	28	.123	.51983	237	.728	32
29	.44594	.822	.072	.506	31	29	.149	.52020	1.9223	.715	31
30	.620	.858	.057	.89493	30	30	.175	.057	210	.701	30
31	.646	.894	.042	.480	29	31	.201	.094	196	.88688	29
32	.672	.931	.028	.467	28	32	.226	.131	183	.674	28
33	.698	.49967	2.0013	.454	27	33	.46252	.168	169	.661	27
34	.724	.50004	1.9999	.441	26	34	.278	.205	155	.647	26
35	.750	.040	.984	.428	25	35	.304	.242	142	.634	25
36	.44776	.076	.970	.415	24	36	.330	.279	128	.620	24
37	.802	.113	.955	.402	23	37	.355	.316	1.9115	.607	23
38	.828	.149	.941	.389	22	38	.381	.52353	101	.88593	22
39	.854	.185	.920	.376	21	39	.407	.390	088	.580	21
40	.880	.222	.912	.89303	20	40	.433	.427	.074	.566	20
41	.906	.258	.897	.350	19	41	.458	.464	.061	.553	19
42	.932	.295	.883	.337	18	42	.46484	.501	.047	.539	18
43	.958	.50331	1.9868	.324	17	43	.510	.538	.034	.526	17
44	.44084	.368	.854	.311	16	44	.536	.575	.020	.512	16
45	.45010	.404	.840	.298	15	45	.561	.613	1.9007	.88499	15
46	.036	.441	.825	.285	14	46	.587	.650	1.8993	.485	14
47	.062	.477	.811	.272	13	47	.613	.52687	.980	.472	13
48	.088	.514	.797	.259	12	48	.639	.724	.967	.458	12
49	.114	.550	.782	.245	11	49	.664	.761	.953	.445	11
50	.140	.587	.768	.89232	10	50	.690	.798	.940	.431	10
51	.166	.623	.754	.219	9	51	.46716	.836	.927	.417	9
52	.45192	.50660	1.9740	.206	8	52	.742	.873	.913	.404	8
53	.218	.696	.725	.193	7	53	.767	.910	1.8900	.88390	7
54	.243	.733	.711	.180	6	54	.793	.947	.887	.377	6
55	.269	.769	.697	.167	5	55	.819	.52985	.873	.363	5
56	.295	.806	.683	.153	4	56	.844	.53022	.860	.349	4
57	.321	.843	.669	.140	3	57	.870	.059	.847	.336	3
58	.347	.879	.654	.127	2	58	.896	.096	.834	.322	2
59	.373	.916	.640	.114	1	59	.921	.134	.820	.308	1
60	.45399	.50593	1.9626	.89101	0	60	.46947	.53171	1.8807	.88295	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'

63°

READ UP

62°

SOLUTION OF EQUATIONS

32°

READ DOWN

33°

	sin	tan	cot	cos		sin	tan	cot	cos		
0	.52992	.62457	1.6003	.84805	60	0	.54464	.64941	1.5399	.83867	60
1	.53017	527	1.5993	.789	59	1	488	.64982	.389	.851	59
2	041	568	983	.774	58	2	513	.65024	.379	.835	58
3	066	608	972	.759	57	3	537	.665	.369	.819	57
4	091	649	962	.743	56	4	561	.666	.359	.804	56
5	115	689	952	.728	55	5	586	148	.350	.788	55
6	140	.62730	941	.712	54	6	610	.189	.340	.772	54
7	164	770	931	.697	53	7	.54635	.231	.330	.83756	53
8	189	811	921	.686	52	8	659	.272	1.5320	.740	52
9	.53214	852	911	.666	51	9	683	.65314	.311	.721	51
10	235	892	1.5900	.650	50	10	708	.355	.301	.708	50
11	263	933	890	.635	49	11	732	.397	.291	.692	49
12	288	.62973	880	.619	48	12	756	.438	.282	.676	48
13	312	.63014	869	.601	47	13	781	.450	.272	.660	47
14	337	.055	859	.588	46	14	805	.521	.262	.645	46
15	361	.095	849	.84573	45	15	.54829	.563	.253	.83629	45
16	.53386	136	839	.557	44	16	854	.604	1.5243	.613	44
17	411	177	829	.542	43	17	878	.65646	.233	.597	43
18	435	217	818	.526	42	18	902	.688	.224	.581	42
19	460	258	808	.511	41	19	927	.729	.214	.565	41
20	484	299	1.5795	.495	40	20	951	.771	.204	.549	40
21	509	.63340	788	.480	39	21	975	.813	.195	.533	39
22	534	380	778	.464	38	22	.54999	.854	.185	.517	38
23	558	421	768	.84448	37	23	.55024	.896	.175	.83501	37
24	.53563	462	757	.433	36	24	048	.1.5166	.485	.36	
25	607	503	747	.417	35	25	072	.65980	.156	.460	35
26	632	544	737	.402	34	26	097	.66021	.147	.453	34
27	656	584	727	.386	33	27	121	.063	.137	.437	33
28	681	625	717	.370	32	28	145	.105	.127	.421	32
29	705	.63666	707	.355	31	29	169	.147	.118	.405	31
30	730	707	1.5697	.84339	30	30	.55194	.189	.108	.380	30
31	754	748	.687	.324	29	31	218	.230	.099	.83373	29
32	.53779	789	.677	.308	28	32	242	.272	1.5089	.356	28
33	804	830	.667	.292	27	33	266	.314	.080	.340	27
34	828	871	.657	.277	26	34	291	.66356	.070	.321	26
35	853	912	.647	.261	25	35	315	.398	.061	.308	25
36	877	953	.637	.245	24	36	339	.440	.051	.292	24
37	902	.63994	627	.230	23	37	.55363	.482	.042	.276	23
38	926	.64035	617	.84214	22	38	388	.524	.032	.260	22
39	951	.076	607	.198	21	39	412	.566	.023	.83244	21
40	.53975	117	1.5597	.182	20	40	438	.008	.013	.228	20
41	.54000	158	.587	.167	19	41	460	.66050	1.5004	.212	19
42	024	199	.577	.151	18	42	484	.692	1.4994	.195	18
43	049	240	.567	.135	17	43	509	.734	.985	.179	17
44	073	281	.557	.120	16	44	533	.776	.975	.163	16
45	097	.64322	.547	.84104	15	45	.55557	.818	.966	.147	15
46	122	303	.537	.088	14	46	581	.860	.957	.131	14
47	146	404	.527	.072	13	47	603	.902	.947	.83115	13
48	171	446	.517	.057	12	48	630	.944	.938	.098	12
49	195	487	.507	.041	11	49	.654	.66986	.928	.082	11
50	.54220	628	1.5497	.025	10	50	678	.67028	.919	.066	10
51	244	569	.487	.84009	9	51	702	.071	1.4910	.050	9
52	269	.64610	.477	.83994	8	52	.55726	.113	.900	.034	8
53	293	652	.468	.978	7	53	.750	.153	.891	.017	7
54	317	.093	.455	.962	6	54	.775	.197	.882	.83001	6
55	342	734	.448	.946	5	55	.790	.230	.872	.82985	5
56	366	775	.438	.930	4	56	.823	.282	.863	.909	4
57	391	817	.428	.915	3	57	.847	.324	.854	.953	3
58	415	858	.418	.899	2	58	.871	.366	.844	.936	2
59	440	899	.408	.883	1	59	.895	.409	.835	.920	1
60	.54464	.64041	1.5399	.83567	0	60	.55919	.67451	1.4820	.82004	0
	cos	cot	tan	sin			cos	cot	tan	sin	

57°

READ UP

56°

TRIGONOMETRIC FUNCTIONS

267

34°

READ DOWN

35°

'	sin	tan	cot	cos		'	sin	tan	cot	cos		
0	.55919	.67451	1.4826	.82904	60	0	.57358	.70021	1.4281	.81915	60	
1	.943	.493	.816	.887	59	1	.381	.064	.273	.899	59	
2	.968	.536	.807	.871	58	2	.405	.107	.264	.882	58	
3	.55992	.578	.798	.855	57	3	.429	.151	.255	.865	57	
4	.56016	.620	.788	.839	56	4	.453	.194	.246	.848	56	
5		.040	.663	.779	.822	55	5	.477	.238	.237	.832	55
6		.064	.67705	.770	.806	54	6	.501	.281	.229	.815	54
7		.088	.748	.761	.790	53	7	.524	.70325	.220	.81798	53
8		.112	.790	.751	.773	52	8	.548	.368	1.4211	.782	52
9		.136	.832	.742	.82757	51	9	.57572	.412	.202	.765	51
10		.160	.875	1.4733	.741	50	10	.596	.455	.193	.748	50
11		.184	.917	.724	.724	49	11	.619	.499	.185	.731	49
12		.56208	.67960	.715	.708	48	12	.643	.542	.176	.714	48
13		.232	.68002	.705	.692	47	13	.667	.586	.167	.698	47
14		.256	.045	.696	.675	46	14	.691	.629	.158	.681	46
15		.280	.088	.687	.659	45	15	.715	.70673	.150	.81664	45
16		.305	.130	.678	.643	44	16	.738	.717	1.4141	.647	44
17		.329	.173	.669	.626	43	17	.762	.760	.132	.631	43
18		.353	.215	.659	.82610	42	18	.57786	.804	.124	.614	42
19		.377	.258	.650	.593	41	19	.810	.848	.115	.597	41
20		.56401	.301	1.4641	.577	40	20	.833	.891	.106	.580	40
21		.425	.68343	.632	.561	39	21	.857	.935	.097	.563	39
22		.449	.386	.623	.544	38	22	.881	.70979	.089	.546	38
23		.473	.429	.614	.528	37	23	.904	.71023	.080	.81530	37
24		.497	.471	.605	.511	36	24	.928	.066	1.4071	.513	36
25		.521	.514	.596	.495	35	25	.952	.110	.063	.496	35
26		.545	.557	.586	.478	34	26	.976	.154	.054	.479	34
27		.569	.600	.577	.82462	33	27	.57999	.198	.045	.462	33
28		.56593	.68642	.568	.446	32	28	.58023	.242	.037	.445	32
29		.617	.685	.559	.429	31	29	.047	.285	.028	.428	31
30		.641	.728	1.4550	.413	30	30	.070	.71329	.019	.412	30
31		.665	.771	.541	.396	29	31	.094	.373	.011	.81395	29
32		.689	.814	.532	.380	28	32	.118	.417	1.4002	.378	28
33		.713	.857	.523	.363	27	33	.141	.461	1.3994	.361	27
34		.736	.900	.514	.347	26	34	.165	.505	.985	.344	26
35		.760	.942	.505	.330	25	35	.189	.549	.976	.327	25
36		.56784	.68935	.496	.82314	24	36	.58212	.593	.968	.310	24
37		.808	.69028	.487	.297	23	37	.236	.637	.959	.293	23
38		.832	.071	.478	.281	22	38	.260	.71681	.951	.276	22
39		.856	.114	.469	.264	21	39	.283	.725	.942	.81259	21
40		.880	.157	1.4460	.248	20	40	.307	.769	.934	.242	20
41		.804	.200	.451	.231	19	41	.330	.813	.925	.225	19
42		.928	.243	.442	.214	18	42	.354	.857	1.3916	.208	18
43		.952	.286	.433	.198	17	43	.378	.901	.908	.191	17
44		.56976	.69329	.424	.181	16	44	.58401	.946	.899	.174	16
45		.57000	.372	.415	.165	15	45	.425	.71990	.891	.157	15
46		.024	.416	.406	.82148	14	46	.449	.72034	.882	.140	14
47		.047	.459	.397	.132	13	47	.472	.078	.874	.81123	13
48		.071	.502	.388	.115	12	48	.496	.122	.865	.106	12
49		.095	.545	.379	.098	11	49	.519	.167	.857	.089	11
50		.119	.588	1.4370	.082	10	50	.543	.211	.848	.072	10
51		.143	.631	.361	.065	9	51	.567	.255	1.3840	.055	9
52		.167	.69675	.352	.048	8	52	.58590	.299	.831	.038	8
53		.57191	.718	.344	.032	7	53	.614	.72344	.823	.021	7
54		.215	.761	.335	.82015	6	54	.637	.388	.814	.81004	6
55		.238	.804	.326	.81999	5	55	.661	.432	.806	.80987	5
56		.262	.847	.317	.982	4	56	.684	.477	.798	.970	4
57		.286	.891	.308	.965	3	57	.708	.521	.789	.953	3
58		.310	.934	.299	.949	2	58	.731	.565	.781	.936	2
59		.334	.69977	.290	.932	1	59	.755	.610	.772	.919	1
60		.57358	.70021	1.4281	.81915	0	60	.58779	.72654	1.3764	.80902	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'	

55°

READ UP

54°

SOLUTION OF EQUATIONS

36°

READ DOWN

37°

	sin	tan	cot	cos			sin	tan	cot	cos	
0	.58779	.72654	1.3764	.80902	60	0	.60182	.75355	1.3270	.79364	60
1	.802	.699	.755	.895	59	1	.205	.401	.252	.816	59
2	.826	.743	.747	.867	58	2	.228	.447	.251	.829	58
3	.849	.788	.739	.850	57	3	.251	.492	.246	.811	57
4	.873	.832	.730	.833	56	4	.274	.538	.238	.793	56
5	.896	.877	.722	.816	55	5	.298	.584	.230	.776	55
6	.920	.921	.713	.792	54	6	.321	.629	.222	.758	54
7	.943	.72966	.705	.80782	53	7	.344	.75675	.214	.741	53
8	.967	.73010	.697	.765	52	8	.367	.721	.206	.723	52
9	.55990	.055	.688	.748	51	9	.60390	.767	1.3198	.706	51
10	.59014	100	1.3680	.730	50	10	.414	.812	.190	.70688	50
11	.037	144	.672	.713	49	11	.437	.858	.182	.671	49
12	.061	189	.663	.696	48	12	.460	.901	.175	.653	48
13	.084	234	.655	.679	47	13	.483	.950	.167	.635	47
14	.108	278	.647	.662	46	14	.506	.75996	.159	.618	46
15	.131	.73323	.638	.80644	45	15	.529	.76012	.151	.600	45
16	.154	.368	.630	.627	44	16	.553	.958	.143	.583	44
17	.178	.413	.622	.610	43	17	.60576	.134	1.3135	.565	43
18	.201	.457	.613	.593	42	18	.590	.180	.127	.547	42
19	.58923	.592	.605	.578	41	19	.622	.224	.119	.530	41
20	.245	.547	1.3597	.558	40	20	.645	.272	.111	.79512	40
21	.272	.592	.588	.541	39	21	.668	.318	.103	.494	39
22	.295	.73637	.580	.80524	38	22	.691	.361	.095	.477	38
23	.318	.681	.572	.507	37	23	.714	.410	.087	.459	37
24	.342	.726	.564	.489	36	24	.738	.456	.079	.441	36
25	.365	.771	.555	.472	35	25	.761	.76502	.072	.424	35
26	.389	.816	.547	.455	34	26	.C0784	.548	1.3001	.406	34
27	.59412	.861	.539	.438	33	27	.807	.594	.056	.388	33
28	.436	.906	.531	.420	32	28	.830	.610	.048	.371	32
29	.459	.951	.522	.403	31	29	.853	.656	.010	.353	31
30	.482	.72996	1.3514	.80380	30	30	.876	.733	.032	.79335	30
31	.506	.74011	.506	.368	29	31	.890	.779	.024	.318	29
32	.529	.056	.498	.351	28	32	.922	.825	.017	.300	28
33	.552	.131	.490	.334	27	33	.915	.871	.009	.282	27
34	.576	.176	.481	.316	26	34	.908	.918	1.3001	.261	26
35	.59399	.221	.473	.299	25	35	.60901	.76964	1.2993	.247	25
36	.622	.267	.465	.282	24	36	.61015	.77010	.985	.229	24
37	.646	.312	.457	.80261	23	37	.639	.057	.977	.211	23
38	.669	.74357	.419	.217	22	38	.661	.103	.970	.193	22
39	.693	.402	.410	.230	21	39	.684	.149	.962	.170	21
40	.716	.447	1.3432	.212	20	40	.107	.196	.954	.79155	20
41	.739	.492	.421	.195	19	41	.130	.212	.946	.140	19
42	.763	.538	.416	.178	18	42	.153	.289	.038	.122	18
43	.59786	.583	.408	.160	17	43	.176	.77335	1.2931	.105	17
44	.809	.628	.400	.143	16	44	.61199	.382	.923	.087	16
45	.832	.74674	.392	.80125	15	45	.222	.428	.915	.069	15
46	.856	.719	.384	.108	14	46	.245	.475	.907	.051	14
47	.879	.764	.375	.091	13	47	.268	.521	.900	.033	13
48	.902	.810	.367	.073	12	48	.291	.568	.802	.79016	12
49	.926	.855	.359	.056	11	49	.314	.615	.881	.78308	11
50	.949	.900	1.3351	.038	10	50	.337	.77661	.876	.980	10
51	.972	.016	.313	.021	9	51	.360	.708	.869	.962	9
52	.59995	.74991	.335	.80003	8	52	.61383	.751	1.2861	.911	8
53	.60010	.75037	.327	.79986	7	53	.406	.801	.853	.926	7
54	.012	.082	.319	.968	6	54	.429	.818	.846	.905	6
55	.065	.128	.311	.951	5	55	.451	.893	.838	.891	5
56	.089	.173	.303	.934	4	56	.474	.941	.830	.873	4
57	.112	.210	.295	.916	3	57	.497	.77988	.822	.855	3
58	.135	.264	.287	.899	2	58	.520	.78035	.815	.837	2
59	.158	.310	.278	.881	1	59	.543	.882	.807	.819	1
60	.60182	.75355	1.3270	.79564	0	60	.61566	.78120	1.2799	.78801	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

53°

READ UP

52°

TRIGONOMETRIC FUNCTIONS

269

38°

READ DOWN

39°

'	sin	tan	cot	cos		'	sin	tan	cot	cos	
0	.61566	.78129	1.2799	.78801	60	0	.62932	.80978	1.2349	.77715	60
1	.589	175	792	783	59	1	.955	.81027	342	.696	59
2	.612	222	784	765	58	2	.62977	.075	334	.678	58
3	.635	269	776	747	57	3	.63000	123	327	.660	57
4	.658	316	769	729	56	4	.022	171	320	.641	56
5	.681	363	761	711	55	5	.045	220	312	.623	55
6	.704	410	753	694	54	6	.068	268	305	.605	54
7	.726	457	746	676	53	7	.090	316	298	.586	53
8	.749	504	738	.78658	52	8	.113	364	290	.77568	52
9	.61772	.78551	731	.640	51	9	.135	413	283	.550	51
10	.795	598	1.2723	.622	50	10	.63158	461	1.2276	.531	50
11	.818	645	715	.604	49	11	.180	.81510	268	.513	49
12	.841	692	708	.586	48	12	.203	.558	.261	.494	48
13	.864	739	700	.568	47	13	.225	.606	.254	.476	47
14	.887	786	693	.550	46	14	.248	.655	.247	.458	46
15	.909	834	685	.78532	45	15	.271	703	.239	.439	45
16	.932	881	677	.514	44	16	.293	.752	.232	.77421	44
17	.955	928	670	.496	43	17	.63316	.800	.225	.402	43
18	.61978	.78975	662	.478	42	18	.338	.849	.218	.384	42
19	.62001	.79022	655	.460	41	19	.361	.898	.210	.366	41
20	.024	.070	1.2647	.442	40	20	.383	.946	1.2203	.347	40
21	.046	.117	640	.424	39	21	.406	.81995	.196	.329	39
22	.069	.164	632	.78405	38	22	.428	.82044	.189	.310	38
23	.092	.212	624	.387	37	23	.451	.092	.181	.77292	37
24	.115	.259	617	.369	36	24	.63473	.141	.174	.273	36
25	.138	.306	609	.351	35	25	.496	.190	.167	.255	35
26	.160	.354	602	.333	34	26	.518	.238	.160	.236	34
27	.62183	.401	.594	.315	33	27	.540	.287	.153	.218	33
28	.206	.449	.587	.297	32	28	.563	.336	.145	.199	32
29	.229	.79496	.579	.78279	31	29	.585	.385	.138	.181	31
30	.251	.544	1.2572	.261	30	30	.608	.434	1.2131	.77162	30
31	.274	.591	.564	.243	29	31	.63630	.82483	.124	.144	29
32	.297	.639	.557	.225	28	32	.653	.531	.117	.125	28
33	.320	.686	.549	.206	27	33	.675	.580	.109	.107	27
34	.342	.734	.542	.188	26	34	.698	.629	.102	.088	26
35	.62365	.781	.534	.170	25	35	.720	.678	.095	.070	25
36	.388	.829	.527	.78152	24	36	.742	.727	.088	.051	24
37	.411	.877	.519	.134	23	37	.765	.776	.081	.033	23
38	.433	.924	.512	.116	22	38	.787	.825	.074	.77014	22
39	.456	.79972	.504	.098	21	39	.63810	.874	.066	.76996	21
40	.479	.80020	1.2497	.079	20	40	.832	.923	1.2059	.977	20
41	.502	.067	.489	.061	19	41	.854	.82972	.052	.959	19
42	.524	.115	.482	.043	18	42	.877	.83022	.045	.940	18
43	.547	.163	.475	.025	17	43	.899	.071	.038	.921	17
44	.62570	.211	.467	.78007	16	44	.922	.120	.031	.903	16
45	.592	.258	.460	.77988	15	45	.944	.169	.024	.884	15
46	.615	.306	.452	.970	14	46	.966	.218	.017	.76866	14
47	.638	.354	.445	.952	13	47	.63989	.268	.009	.847	13
48	.660	.402	.437	.934	12	48	.64011	.317	1.2002	.828	12
49	.683	.450	.430	.916	11	49	.033	.366	1.1995	.810	11
50	.706	.80498	1.2423	.897	10	50	.056	.415	.988	.791	10
51	.728	.546	.415	.870	9	51	.078	.83465	.981	.772	9
52	.62751	.594	.408	.77861	8	52	.100	.514	.974	.754	8
53	.774	.642	.401	.843	7	53	.123	.564	.967	.76735	7
54	.796	.690	.393	.824	6	54	.64145	.613	.960	.717	6
55	.819	.738	.386	.806	5	55	.167	.662	.953	.698	5
56	.842	.786	.378	.788	4	56	.190	.712	.946	.679	4
57	.864	.834	.371	.769	3	57	.212	.761	.939	.661	3
58	.887	.882	.364	.751	2	58	.234	.811	.932	.642	2
59	.909	.930	.356	.733	1	59	.256	.860	.925	.623	1
60	.62932	.80978	1.2349	.77715	0	60	.64279	.83910	1.1918	.76604	0
	cos	cot	tan	sin	'		cos	cot	tan	sin	'

51°

READ UP

50°

SOLUTION OF EQUATIONS

40°

READ DOWN

41°

	sin	tan	cot	cos		sin	tan	cot	cos		
	cos	cot	tan	sin		cos	cot	tan	sin		
0	.64279	.83910	1.19118	.76604	60	0	.65606	.86929	1.1504	.75471	60
1	.301	.83950	910	.586	59	1	.628	.86980	.497	.452	59
2	.323	.84009	903	.567	58	2	.650	.87031	.490	.433	58
3	.346	.059	896	.548	57	3	.672	.082	.483	.414	57
4	.368	103	889	.530	56	4	.694	.133	.477	.395	56
5	.390	158	882	.511	55	5	.716	.184	.470	.375	55
6	.412	205	875	.492	54	6	.738	.236	.463	.356	54
7	.435	258	868	.473	53	7	.759	.287	.456	.337	53
8	.64457	307	861	.76455	52	8	.781	.338	.450	.75318	52
9	.479	357	854	.436	51	9	.65803	.389	.413	.299	51
10	.601	407	1.1847	.417	50	10	.825	.411	1.1436	.280	50
11	.524	.84457	840	.398	49	11	.847	.87492	.430	.261	49
12	.546	507	833	.380	48	12	.869	.543	.423	.241	48
13	.568	.556	826	.361	47	13	.891	.595	.416	.222	47
14	.590	606	819	.342	46	14	.913	.646	.410	.203	46
15	.612	.656	812	.323	45	15	.935	.698	.403	.184	45
16	.64635	706	806	.76304	44	16	.956	.749	.396	.75165	44
17	.657	.756	799	.296	43	17	.65978	.801	.389	.146	43
18	.679	.806	792	.267	42	18	.66000	.852	.383	.126	42
19	.701	.856	785	.218	41	19	.022	.904	.376	.107	41
20	.723	.906	1.1778	.229	40	20	.044	.87955	1.1369	.088	40
21	.746	.84956	771	.210	39	21	.066	.88000	.363	.069	39
22	.768	.85006	764	.192	38	22	.088	.059	.356	.050	38
23	.790	.057	757	.173	37	23	.109	.110	.349	.030	37
24	.64812	107	.750	.76154	36	24	.131	.162	.343	.75011	36
25	.834	.157	743	.135	35	25	.66153	.214	.336	.74992	35
26	.856	.207	736	.116	34	26	.175	.265	.329	.973	34
27	.878	.257	729	.097	33	27	.197	.317	.323	.953	33
28	.901	.303	722	.078	32	28	.218	.369	.316	.934	32
29	.923	.358	715	.059	31	29	.240	.421	.310	.915	31
30	.945	.405	1.1708	.041	30	30	.262	.88473	1.1303	.896	30
31	.967	.85458	702	.022	29	31	.284	.524	.296	.876	29
32	.64989	.503	.605	.76003	28	32	.66306	.576	.290	.857	28
33	.65011	.559	.688	.75984	27	33	.327	.628	.233	.838	27
34	.033	.609	.631	.065	26	34	.349	.680	.276	.74818	26
35	.055	.660	.674	.946	25	35	.371	.732	.270	.799	25
36	.077	.710	.667	.927	24	36	.393	.784	.263	.780	24
37	.100	.761	.660	.905	23	37	.414	.836	.257	.760	23
38	.122	.811	.653	.889	22	38	.436	.888	.250	.741	22
39	.144	.862	.647	.870	21	39	.66458	.940	.213	.722	21
40	.166	.912	1.1640	.851	20	40	.480	.88902	1.1237	.703	20
41	.188	.85963	633	.832	19	41	.501	.89045	.230	.683	19
42	.65210	.86014	626	.75813	18	42	.523	.097	.224	.664	18
43	.232	.061	619	.794	17	43	.545	.149	.217	.74644	17
44	.254	.115	612	.775	16	44	.566	.201	.211	.623	16
45	.276	.166	606	.756	15	45	.588	.253	.204	.606	15
46	.298	.216	.599	.735	14	46	.66610	.306	.197	.586	14
47	.320	.267	.592	.719	13	47	.632	.358	.191	.567	13
48	.342	.318	.585	.700	12	48	.653	.410	.184	.548	12
49	.364	.369	.578	.680	11	49	.675	.463	.178	.528	11
50	.65356	.419	1.1571	.661	10	50	.697	.89515	1.1171	.509	10
51	.408	.86470	565	.75642	9	51	.718	.567	.165	.71439	9
52	.430	.521	.558	.623	8	52	.740	.620	.158	.470	8
53	.452	.572	.551	.604	7	53	.68762	.672	.152	.451	7
54	.474	.623	.544	.585	6	54	.783	.725	.145	.431	6
55	.496	.674	.538	.506	5	55	.805	.777	.139	.412	5
56	.518	.725	.531	.547	4	56	.827	.830	.132	.392	4
57	.540	.776	.524	.528	3	57	.848	.893	.126	.373	3
58	.562	.827	.517	.509	2	58	.870	.935	.119	.353	2
59	.584	.878	.510	.490	1	59	.891	.89988	.113	.334	1
60	.65006	.86929	1.1504	.75471	0	60	.66913	.90010	1.1106	.74314	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

49°

READ UP

48°

TRIGONOMETRIC FUNCTIONS

271

42°

READ DOWN

43°

	sin	tan	cot	cos			sin	tan	cot	cos	
0	.66913	.90040	1.1106	.74314	60	0	.68200	.93252	1.0724	.73135	60
1	.935	.093	100	295	59	1	.221	.306	.717	.116	59
2	.956	.146	.093	276	58	2	.242	.360	.711	.096	58
3	.978	.199	.087	256	57	3	.264	.415	.705	.076	57
4	.66999	.251	.080	237	56	4	.285	.469	.699	.056	56
5	.67021	.304	.074	217	55	5	.306	.524	.692	.036	55
6	.043	.357	.067	198	54	6	.327	.93578	.686	.73016	54
7	.064	.410	.061	178	53	7	.349	.633	.680	.72996	53
8	.086	.463	1.1054	.74159	52	8	.370	.688	.674	.976	52
9	.107	.90516	.048	139	51	9	.68391	.742	.668	.957	51
10	.129	.569	.041	120	50	10	.412	.797	1.0661	.937	50
11	.151	.621	.035	100	49	11	.434	.852	.655	.917	49
12	.172	.674	.028	080	48	12	.455	.906	.649	.897	48
13	.67194	.727	.022	061	47	13	.476	.93961	.643	.877	47
14	.215	.781	.016	041	46	14	.497	.94016	.637	.857	46
15	.237	.834	.009	022	45	15	.518	.071	.630	.72837	45
16	.258	.887	1.1003	.74002	44	16	.539	.125	.624	.817	44
17	.280	.940	1.0996	.73983	43	17	.561	.180	.618	.797	43
18	.301	.90993	.990	.963	42	18	.68582	.235	.612	.777	42
19	.323	.91046	.983	.944	41	19	.603	.290	.606	.757	41
20	.67344	.099	.977	.924	40	20	.624	.345	1.0599	.737	40
21	.366	.153	.971	.904	39	21	.645	.400	.593	.717	39
22	.387	.206	.964	.885	38	22	.666	.455	.587	.697	38
23	.409	.259	.958	.865	37	23	.688	.94510	.581	.72677	37
24	.430	.313	.951	.846	36	24	.709	.565	.575	.657	36
25	.452	.366	1.0945	.73826	35	25	.730	.620	.569	.637	35
26	.473	.419	.939	.806	34	26	.751	.676	.562	.617	34
27	.495	.473	.932	.787	33	27	.68772	.731	.556	.597	33
28	.67516	.91526	.926	.767	32	28	.793	.786	.550	.577	32
29	.538	.580	.919	.747	31	29	.814	.841	.544	.557	31
30	.559	.633	.913	.728	30	30	.835	.896	1.0538	.637	30
31	.580	.687	.907	.708	29	31	.857	.94952	.532	.517	29
32	.602	.740	.900	.688	28	32	.878	.95007	.526	.72497	28
33	.623	.794	.894	.669	27	33	.899	.062	.519	.477	27
34	.645	.847	1.0888	.73649	26	34	.920	.118	.513	.457	26
35	.67666	.901	.881	.629	25	35	.941	.173	.507	.437	25
36	.688	.91955	.875	.610	24	36	.962	.229	.501	.417	24
37	.709	.92008	.869	.590	23	37	.68983	.284	.495	.397	23
38	.730	.062	.862	.570	22	38	.69004	.340	.489	.377	22
39	.752	.116	.856	.551	21	39	.025	.395	.483	.357	21
40	.773	.170	.850	.531	20	40	.046	.451	1.0477	.337	20
41	.795	.224	.843	.511	19	41	.067	.95506	.470	.72317	19
42	.816	.277	1.0837	.73491	18	42	.088	.562	.464	.297	18
43	.67837	.331	.831	.472	17	43	.109	.618	.458	.277	17
44	.850	.385	.824	.452	16	44	.130	.673	.452	.257	16
45	.880	.439	.818	.432	15	45	.69151	.729	.446	.236	15
46	.901	.92493	.812	.413	14	46	.172	.785	.440	.216	14
47	.923	.547	.805	.393	13	47	.193	.841	.434	.196	13
48	.944	.601	.799	.373	12	48	.214	.897	.428	.72176	12
49	.965	.655	.793	.353	11	49	.235	.95952	.422	.156	11
50	.67087	.709	.786	.333	10	50	.256	.96008	1.0416	.136	10
51	.68008	.763	1.0780	.73314	9	51	.277	.064	.410	.116	9
52	.029	.817	.774	.294	8	52	.69298	.120	.404	.095	8
53	.051	.872	.768	.274	7	53	.319	.176	.398	.075	7
54	.072	.926	.761	.254	6	54	.340	.232	.392	.055	6
55	.093	.92980	.755	.234	5	55	.361	.288	.385	.035	5
56	.115	.93034	.749	.215	4	56	.382	.344	.379	.72015	4
57	.136	.088	.742	.195	3	57	.403	.400	.373	.71995	3
58	.157	.143	.736	.175	2	58	.424	.457	.367	.974	2
59	.179	.197	.730	.155	1	59	.445	.513	.361	.954	1
60	.68200	.93252	1.0724	.73135	0	60	.69466	.96569	I.0355	.71934	0
	cos	cot	tan	sin	,		cos	cot	tan	sin	,

47°

READ UP

46°

SOLUTION OF EQUATIONS

44° READ DOWN					44°						
	sin	tan	cot	cos		sin	tan	cot	cos		
0	.69466	.96569	1.0355	.71934	.60	30	.091	.270	1.0170	.325	.30
1	.487	.625	.349	.914	.50	31	.112	.327	.170	.305	.29
2	.508	.681	.343	.894	.58	32	.132	.384	.164	.284	.28
3	.520	.738	.337	.873	.57	33	.153	.441	.158	.264	.27
4	.549	.791	.331	.853	.56	34	.70174	.98499	.152	.243	.20
5	.570	.850	.325	.833	.55	35	.195	.556	.147	.223	.25
6	.591	.907	.319	.813	.54	36	.215	.613	.141	.71203	.24
7	.69612	.96963	.313	.792	.53	37	.236	.671	.135	.182	.23
8	.633	.97020	.307	.772	.52	38	.257	.728	.129	.162	.22
9	.654	.076	.301	.71752	.51	39	.277	.786	.123	.141	.21
10	.675	.133	L 0295	.732	.50	40	.298	.843	L 0117	.121	.20
11	.696	.189	.289	.711	.49	41	.319	.901	.111	.100	.19
12	.717	.246	.283	.691	.48	42	.70339	.98958	.105	.080	.18
13	.737	.302	.277	.671	.47	43	.360	.99016	.099	.059	.17
14	.758	.359	.271	.650	.46	44	.381	.073	.091	.039	.16
15	.779	.416	.265	.630	.45	45	.401	.131	.088	.71019	.15
16	.69500	.07472	.259	.610	.44	46	.422	.189	.082	.70998	.14
17	.821	.529	.253	.590	.43	47	.443	.247	.076	.978	.13
18	.842	.586	.247	.71560	.42	48	.463	.304	.070	.957	.12
19	.862	.643	.241	.549	.41	49	.484	.362	.064	.937	.11
20	.883	.700	L 0235	.529	.40	50	.505	.420	L 0058	.916	.10
21	.904	.756	.230	.508	.39	51	.70525	.99178	.052	.896	.9
22	.925	.813	.224	.483	.38	52	.546	.536	.047	.875	.8
23	.946	.870	.218	.463	.37	53	.567	.594	.041	.70855	.7
24	.966	.927	.212	.447	.36	54	.587	.652	.035	.831	.6
25	.69987	.97984	.206	.427	.35	55	.608	.710	.029	.813	.5
26	.70008	.98041	.200	.407	.34	56	.628	.768	.023	.793	.4
27	.029	.098	.194	L 1386	.33	57	.649	.826	.017	.772	.3
28	.019	.155	.188	.306	.32	58	.670	.884	.012	.752	.2
29	.070	.213	.182	.345	.31	59	.690	.99442	.006	.731	.1
30	.091	.270	L 0176	.325	.30	60	.70711	L 0.0000	L 0.0000	.70711	0
	cos	cot	tan	sin	/		cos	cot	tan	sin	/

45° READ UP

LOGARITHM TABLES

No.	Log.								
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643153	64	1.806180	84	1.924279
5	0.699870	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819344	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869323	94	1.973128
15	1.176031	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880914	96	1.982271
17	1.230449	37	1.568202	57	1.755975	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892005	98	1.991226
19	1.278754	39	1.591065	59	1.770952	79	1.897627	99	1.995335
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

LOGARITHM TABLES

273

N.	0	1	2	3	4	5	6	7	8	9
100	000000	000434	000868	001301	001734	002166	002596	003029	003461	003891
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775
5	021189	021603	022016	022428	022842	023252	023664	024075	4486	4896
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998
J10	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320
5	000698	061075	061452	061829	062206	062582	062958	063333	063709	4083
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
I20	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026
6	100371	100715	101059	101403	101747	102991	102434	102777	103119	3462
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609
I30	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012
5	130334	130655	130977	131298	131619	131939	132200	132580	132900	3219
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
I40	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802
I50	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407
3	4691	4955	5259	5542	5825	6108	6391	6674	6956	7239
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124
9	201397	201670	201943	202216	202488	2761	3033	3303	3577	3848

N. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

SOLUTION OF EQUATIONS

N.	0	-1	2	3	4	5	6	7	8	9
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556
1	6826	7036	7365	7634	7904	8173	8441	8710	8979	9247
2	9515	9783	210051	210319	210586	210853	211121	211389	211654	211921
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	210193
170	210449	230704	230960	231215	231470	231724	231979	232234	232488	232742
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
2	5525	5781	6033	6285	6537	6789	7041	7292	7544	7795
3	8046	8297	8548	8799	9049	9299	9550	9800	210050	240300
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439
1	7673	7918	8158	8398	8637	8877	9116	9355	9594	9833
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806
1	251033	281261	281488	281715	281942	282169	282496	282722	2849	3075
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
5	230035	230257	230480	230702	230925	231147	231369	231591	231813	232034
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
4	9630	9843	310056	310268	310481	310693	310906	311118	311330	311542
5	311754	311966	2177	2389	2600	2812	3023	3234	3445	3656
6	3867	4078	4290	4499	4710	4920	5130	5340	5551	5760
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
8	8063	8272	8481	8690	8898	9106	9314	9522	9730	9938
9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012
210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077
1	4282	4498	4694	4890	5105	5310	5516	5721	5926	6131
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
3	8380	8583	8787	8991	9194	9398	9601	9805	330008	330211
4	330414	330617	330919	331022	331225	331427	331630	331832	2034	2236
5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246
9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225

N.	0	1	2	3	4	5	6	7	8	9
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LOGARITHM TABLES

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N.	0	1	2	3	4	5	6	7	8	9
220	342423	342620	342817	343014	343212	343409	343606	343803	343999	344196
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	350054
4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
9	9835	360025	360215	360404	360593	360783	360972	361161	361350	361539
230	361728	361917	362105	362294	362482	362671	362859	363048	363236	363424
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030
240	380211	380392	380573	380754	380934	381115	381296	381476	381656	381837
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
5	9166	9343	9520	9698	9875	390051	390228	390405	390582	390759
6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766
250	397940	398114	398287	398461	398634	398808	398981	399154	399328	399501
1	9674	9847	400020	400192	400365	400538	400711	400883	401056	401228
2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
7	9933	410102	410271	410440	410609	410771	410946	411114	411283	411451
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	414973	415140	415307	415474	415641	415808	415974	416141	416308	416474
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
3	9956	420121	420286	420451	420616	420781	420945	421110	421275	421439
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
6	4892	5045	5208	5371	5534	5697	5860	6023	6186	6349
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
9	9752	9914	430075	430236	430398	430559	430720	430881	431042	431203
270	431364	431525	431685	431846	432007	432167	432328	432488	432649	432809
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
5	9333	9491	9648	9806	9964	440122	440279	440437	440594	440752
6	440909	441066	441224	441381	441538	1695	1852	2009	2166	2323
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003

N | 0 | 1 | 2 | 3 | 4 || 5 | 6 | 7 | 8 | 9

SOLUTION OF EQUATIONS

N.	0	1	2	3	4	5	6	7	8	9
280	447158	447313	447468	447623	447778	447933	448088	448242	448397	448552
1	8700	8861	9015	9170	9324	9478	9633	9787	9941	450095
2	450249	450403	450557	450711	450865	451018	451172	451326	451479	1633
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
5	4845	4997	6150	6302	6454	6606	6758	6910	6062	6214
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
8	9392	9543	9694	9845	9995	460146	460296	460447	460597	460748
9	160898	161048	461198	461348	461499	1649	1790	1948	2098	2248
290	462398	462548	462697	462847	462997	463146	463296	463445	463594	463744
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
5	9822	9969	470116	470263	470410	470557	470704	470851	470998	471145
6	471292	471438	1685	1732	1878	2025	2171	2318	2464	2610
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
9	5671	5816	6962	6107	6252	6397	6542	6687	6832	6976
300	477121	477266	477411	477555	477700	477844	477989	478133	478278	478422
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
2	480007	480151	480294	480438	480582	480725	480869	481012	481156	481299
3	1443	1580	1723	1872	2016	2159	2302	2445	2588	2731
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
9	9958	490099	490239	490380	490520	490661	490801	490941	491081	491222
310	491362	491502	491642	491782	491922	492062	492201	492341	492481	492621
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
5	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
6	9687	9824	9962	500099	500236	500374	500511	500648	500785	500922
7	501059	501190	501333	1470	1607	1744	1880	2017	2154	2291
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	505150	505286	505421	505557	505693	505828	505964	506099	506234	506370
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
3	9202	9337	9471	9606	9740	9874	510009	510143	510277	510411
4	510545	510670	510813	510947	511081	511215	1349	1482	1616	1750
5	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
8	6874	6006	6139	6271	6403	6535	6668	6800	6932	7064
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	518514	518646	518777	518909	519040	519171	519303	519434	519566	519697
1	9929	9959	520090	520221	520353	520484	520615	520745	520876	521007
2	521118	521269	1400	1530	1661	1792	1922	2053	2183	2314
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
5	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	330072
9	530200	530328	530446	530584	530712	530840	530968	531096	531223	1351

N.	0	1	2	3	4	5	6	7	8	9
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LOGARITHM TABLES

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N	0	1	2	3	4	5	6	7	8	9
340	531479	531607	531734	531862	531990	532117	532245	532372	532500	532627
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693
5	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951
6	9076	9202	9327	9452	9578	9703	9829	9954	540079	540204
7	640329	640455	640580	640705	640830	640955	641080	641205	1330	1454
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944
350	544068	544192	544316	544440	544564	544688	544812	544936	545060	545183
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	550106
5	550228	550351	550473	550595	550717	550840	550962	551084	551206	1328
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182
360	556303	556423	556544	556664	556785	556905	557026	557146	557267	557387
1	7607	7627	7748	7868	7988	8108	8228	8349	8469	8589
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787
3	9907	560026	560146	560265	560385	560504	560624	560743	560863	560982
4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
8	6848	5966	6084	6202	6320	6437	6555	6673	6791	6909
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	568202	568319	568436	568554	568671	568788	568905	569023	569140	509257
1	9374	9491	9608	9725	9842	9959	570076	570193	570309	570426
2	570543	570660	570776	570893	571010	571126	1243	1359	1476	1592
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669
380	579784	579898	580012	580126	580241	580355	580469	580583	580697	580811
1	680925	681039	1153	1267	1381	1495	1608	1722	1836	1950
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
5	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838
9	9950	590061	590173	590284	590396	590507	590619	590730	590842	590953
390	591065	591176	591287	591399	591510	591621	591732	591843	591955	592066
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
5	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774
8	9893	9992	600101	600210	600319	600428	600537	600646	600755	600864
9	6000973	601082	1191	1299	1408	1517	1625	1734	1843	1951

N.	0	1	2	3	4	5	6	7	8	9
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SOLUTION OF EQUATIONS

N.	O	1	2	3	4	5	6	7	8	9
400	602060	601169	602177	604386	602494	602603	602711	602819	602928	603036
1	3144	3253	3361	3469	3577	3686	3794	4902	4010	4118
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	6197
3	5305	5413	5521	5629	5736	5844	5951	6059	6166	6274
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
7	9594	9701	9808	9914	610021	610128	610234	610341	610447	610554
8	610660	610767	610873	610979	1086	1192	1298	1405	1511	1617
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	612784	612890	612996	613102	613207	613313	613419	613525	613630	613736
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
3	6950	6055	6160	6265	6370	6476	6581	6686	6790	6895
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
5	8018	8153	8257	8362	8466	8571	8676	8780	8884	8989
6	9033	9198	9302	9406	9511	9615	9719	9824	9928	620032
7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	623249	623353	623456	623559	623663	623766	623869	623973	624076	624179
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326
7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	633468	633569	633670	633771	633872	633973	634074	634175	634276	634376
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
2	5481	5584	5685	5785	5886	5986	6087	6187	6287	6388
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
5	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
6	9486	9586	9686	9785	9885	9984	640084	640183	640283	640382
7	640481	640581	640680	640779	640879	640978	1077	1177	1276	1376
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	643453	643551	643650	643749	643847	643946	644044	644143	644242	644340
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
5	8300	8488	8555	8653	8750	8848	8945	9043	9140	9237
6	9335	9432	9530	9627	9724	9821	9919	C50016	C50113	C50210
7	650309	650405	650502	650593	650696	650793	650890	0987	1081	1181
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116
450	653213	653309	653405	653502	653598	653695	653791	653888	653984	654080
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
6	8965	9060	9155	9250	9346	9441	9536	9632	9726	9821
7	9916	CC0011	660106	CC0201	CC0296	CC0391	CC0486	CC0581	660676	660771
8	660863	0960	1055	1150	1245	1339	1434	1529	1623	1718
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663

N. O 1 2 3 4 5 6 7 8 9

LOGARITHM TABLES

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N.	0	1	2	3	4	5	6	7	8	9
460	662758	662832	662947	663041	663135	663230	663324	663418	663512	663607
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
7	9317	9410	9503	9596	9689	9782	9875	9967	670060	670153
8	670246	670339	670431	670524	670617	670710	670802	670895	0988	1080
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	672098	672190	672283	672375	672467	672560	672652	672744	672836	672929
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151
480	681241	681332	681422	681513	681603	681693	681784	681874	681964	682055
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107
490	690196	690285	690373	690462	690550	690639	690728	690816	690905	690993
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883
500	698970	699057	699144	699231	699317	699404	699491	699578	699664	699751
1	9838	9924	700011	700098	700184	700271	700358	700444	700531	700617
2	700704	700790	0877	0963	1050	1136	1222	1309	1395	1482
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	707570	707655	707740	707826	707911	707996	708081	708166	708251	708336
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
2	9270	9355	9440	9524	9609	9694	9779	9863	9943	710033
3	710117	710202	710287	710371	710456	710540	710625	710710	710794	0879
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920

SOLUTION OF EQUATIONS

N.	1	2	3	4	5	6	7	8	9
520	716003	716087	716179	716254	716337	716421	716504	716588	716671
1	6838	6921	7004	7088	7171	7254	7338	7421	7504
2	7671	7754	7837	7920	8003	8086	8169	8253	8336
3	8502	8585	8668	8751	8834	8917	9000	9083	9165
4	9331	9414	9497	9580	9663	9745	9828	9911	9994
5	720159	720242	720325	720407	720490	720573	720655	720738	720821
6	0986	1068	1151	1233	1316	1398	1481	1563	1646
7	1811	1893	1975	2058	2140	2222	2305	2387	2469
8	2634	2716	2798	2881	2963	3045	3127	3209	3291
9	3456	3538	3620	3702	3784	3866	3948	4030	4112
530	724276	724358	724440	724522	724604	724685	724767	724849	724931
1	5095	5176	5258	5340	5422	5503	5585	5667	5748
2	5912	5993	6075	6156	6238	6320	6401	6483	6564
3	6721	6803	6886	6972	7053	7134	7216	7297	7379
4	7541	7623	7704	7785	7866	7948	8029	8110	8191
5	8354	8435	8516	8597	8678	8759	8841	8922	9003
6	9165	9246	9327	9408	9489	9570	9651	9732	9813
7	9974	730055	730136	730217	730298	730378	730459	730540	730621
8	730782	0863	0944	1024	1105	1186	1266	1347	1428
9	1589	1669	1750	1830	1911	1991	2072	2152	2233
540	732394	732474	732555	732635	732715	732796	732876	732956	733037
1	3197	3278	3358	3438	3518	3598	3679	3759	3839
2	3999	4079	4160	4240	4320	4400	4480	4560	4640
3	4800	4880	4960	5040	5120	5200	5279	5359	5439
4	5699	5679	5759	5838	5918	5998	6078	6157	6237
5	6397	6476	6556	6635	6715	6795	6874	6954	7034
6	7193	7272	7352	7431	7511	7590	7670	7749	7829
7	7987	8067	8146	8225	8305	8381	8463	8543	8622
8	8781	8860	8939	9018	9097	9177	9256	9335	9414
9	9572	9651	9731	9810	9889	9968	740017	740126	740205
550	740363	740442	740521	740600	740678	740757	740836	740915	740994
1	1152	1230	1309	1388	1467	1546	1624	1703	1782
2	1939	2018	2096	2175	2254	2332	2411	2489	2568
3	2725	2804	2882	2961	3039	3118	3196	3275	3353
4	3510	3588	3667	3745	3823	3902	3980	4058	4136
5	4293	4371	4449	4528	4606	4684	4762	4840	4919
6	5075	5153	5231	5309	5387	5465	5543	5621	5699
7	5855	5933	6011	6089	6167	6245	6323	6401	6479
8	6634	6712	6790	6868	6945	7023	7101	7179	7256
9	7412	7489	7567	7645	7722	7800	7878	7955	8033
560	748188	748266	748344	748421	748498	748576	748653	748731	748808
1	8963	9040	9118	9195	9272	9350	9427	9504	9582
2	9736	9814	9891	9968	750045	750123	750200	750277	750354
3	750508	750586	750663	750740	0817	0894	0971	1048	1125
4	1279	1356	1433	1510	1587	1664	1741	1818	1895
5	2018	2125	2202	2279	2356	2433	2509	2586	2663
6	2816	2893	2970	3047	3123	3200	3277	3353	3430
7	3583	3660	3736	3813	3889	3966	4042	4119	4195
8	4348	4425	4501	4578	4654	4730	4807	4883	4960
9	5112	5189	5265	5341	5417	5494	5570	5646	5722
570	755875	755951	756027	756103	756180	756256	756332	756408	756484
1	6636	6712	6788	6864	6940	7016	7092	7168	7244
2	7396	7472	7548	7624	7700	7775	7851	7927	8003
3	8165	8230	8306	8382	8458	8533	8609	8685	8761
4	8912	8988	9063	9139	9214	9290	9366	9441	9517
5	9668	9743	9819	9894	9970	760045	7C0121	760190	760272
6	760422	760498	760573	760649	760724	0799	0875	0950	1025
7	1176	1251	1326	1402	1477	1552	1627	1702	1778
8	1928	2003	2078	2153	2228	2303	2378	2453	2529
9	2679	2754	2829	2904	2978	3053	3128	3203	3278

N.	0	1	2	3	4	5	6	7	8	9
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LOGARITHM TABLES

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N.	0	1	2	3	4	5	6	7	8	9
580	763428	763503	763578	763653	763727	763802	763877	763952	764027	764101
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778
590	770852	770926	770999	771073	771146	771220	771293	771367	771440	771514
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524
2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
8	3904	3976	4046	4118	4189	4261	4332	4403	4475	4546
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	792392	792462	792532	792602	792672	792742	792812	792882	792952	793022
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961
1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112

SOLUTION OF EQUATIONS

N.	0	1	2	3	4	5	6	7	8	9	
640	806160	806248	806316	806354	806415	806453	806519	806557	806595	806723	806760
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	
4	8880	8933	9021	9088	9156	9223	9290	9358	9425	9492	
5	9500	9627	9694	9762	9829	9896	9964	910031	910098	910165	
6	810233	810300	810367	810434	810501	810569	810636	8703	8770	8837	
7	9304	9371	1039	1106	1173	1240	1307	1374	1441	1508	
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	
660	819544	819610	819676	819741	819807	819873	819939	820004	820070	820136	
1	820201	820267	820333	820399	820464	820530	820595	80661	8727	8792	
2	0858	0924	0990	1055	1120	1186	1251	1317	1382	1448	
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
6	3474	3539	3603	3670	3735	3800	3865	3930	3995	4061	
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	826075	826140	826204	826269	826334	826399	826464	826529	826593	826658	
1	6723	6787	6852	6917	5981	7046	7111	7175	7240	7305	
2	7363	7434	7493	7563	7628	7692	7757	7821	7886	7951	
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	
4	8660	8724	8789	8853	8919	8982	9046	9111	9175	9239	
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
6	9947	530011	830075	830139	830204	830268	830332	830396	830460	830525	
7	530582	0653	0717	0781	0845	0909	0973	1037	1102	1166	
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	832503	832573	832637	832700	832764	832828	832892	832956	833020	833083	
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6893	
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	
690	838849	838912	838975	839038	839101	839164	839227	839290	839352	839415	
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	840043	
2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	
3	0731	0796	0859	0921	0984	1046	1103	1172	1234	1297	
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	

N 0 1 2 3 4 5 6 7 8 9

LOGARITHM TABLES

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N.	0	1	2	3	4	5	6	7	8	9
700	845098	845160	845222	845284	845346	845408	845470	845532	845594	845656
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	851268	851320	851381	851442	851503	851564	851625	851686	851747	851809
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	857332	857393	857453	857513	857574	857634	857694	857755	857815	857875
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
4	9739	9799	9859	9918	9978	860038	860098	860158	860218	860278
5	800338	860398	860458	860518	860578	0637	0697	0757	0817	0877
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	863323	863382	863442	863501	863561	863620	863680	863739	863799	863858
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	869232	869290	869349	869408	869466	869525	869584	869642	869701	869760
1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345
2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003
750	875061	875119	875177	875235	875293	875351	875409	875466	875524	875582
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185
9	880242	880299	880356	880413	880471	880528	0585	0642	0699	0756

N.	0	1	2	3	4	5	6	7	8	9
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SOLUTION OF EQUATIONS

N.	0	1	2	3	4	5	6	7	8	9
760	880814	880871	880928	880985	881042	881099	881156	881213	881271	881328
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
5	3661	3718	3775	3832	3888	3945	4002	4059	4116	4172
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	886491	886547	886604	886660	886716	886773	886829	886885	886942	886998
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
6	9862	9918	9974	800030	890086	890141	890197	890253	890309	890365
7	800421	890477	890533	0589	0645	0700	0756	0812	0868	0924
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	892095	892150	892206	892262	892317	892373	892429	892484	892540	892595
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	897627	897682	897737	897792	897847	897902	897959	898012	898067	898122
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766
4	9821	9875	9930	9985	800039	900094	900149	900203	900258	900312
5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036
800	903090	903144	903199	903253	903307	903361	903416	903470	903524	903578
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
4	5236	5310	5364	5418	5472	5526	5580	5634	5688	5742
5	5795	5850	5904	5958	6012	6066	6119	6173	6227	6281
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
9	7949	8002	8056	8110	8163	8217	8270	8321	8378	8431
810	908485	908539	908592	908646	908699	908753	908807	908860	908914	909967
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037
3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761

N.	0	1	2	3	4	5	6	7	8	9
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LOGARITHM TABLES

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820	913814	913867	913920	913973	914026	914079	914132	914184	914237	914290
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	924279	924331	924383	924434	924486	924538	924589	924641	924693	924744
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389
2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	934498	934549	934599	934650	934700	934751	934801	934852	934902	934953
1	6003	5054	5104	5154	5205	5255	5306	5356	5406	5457
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	939519	969569	939619	939669	939719	939769	939819	939869	939918	939968
1	940018	940068	940118	940168	940218	940267	940317	940367	940417	940467
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
8	3445	3544	3593	3643	3692	3742	3791	3841	3890	3939
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433

N.	0	1	2	3	4	5	6	7	8	9
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SOLUTION OF EQUATIONS

N.	0	1	2	3	4	5	6	7	8	9
880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829
1	9878	9926	9975	995024	950073	950121	950170	950219	950267	950316
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
4	1339	1386	1435	1483	1532	1580	1629	1677	1726	1775
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
7	2792	2841	2890	2938	2987	3034	3083	3131	3180	3228
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194
900	954243	954291	954339	954387	954435	954484	954532	954580	954628	954677
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
3	5689	5736	5784	5832	5880	5928	5976	6024	6072	6120
4	6168	6216	6263	6313	6361	6409	6457	6505	6553	6601
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
8	8080	8134	8181	8229	8277	8325	8373	8421	8468	8516
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	959041	959089	959137	959185	959233	959280	959328	959375	959423	959471
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
2	9905	990442	990090	990138	990185	990233	990281	990328	990376	990423
3	990471	0518	0566	0613	0661	0709	0756	0804	0851	0899
4	0916	0994	1041	1089	1136	1184	1231	1279	1326	1374
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
7	2309	2417	2464	2511	2559	2606	2653	2701	2748	2795
8	2813	2890	2937	2985	3032	3079	3126	3174	3221	3268
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
4	5672	6119	5766	5813	5860	5907	5954	6001	6048	6095
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
8	7549	7595	7642	7688	7735	7782	7829	7875	7922	7969
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	968383	968330	968376	968423	968470	968516	968563	968610	968656	968703
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
3	9832	9928	9975	970021	970064	970114	970161	970207	970254	970300
4	970317	970333	970410	0486	0533	0579	0626	0672	0719	0765
5	0912	0959	0994	0951	0997	1044	1090	1137	1183	1229
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
8	2203	2249	2295	2342	2399	2434	2481	2527	2573	2619
9	2666	2712	2759	2804	2851	2897	2943	2989	3035	3082

N. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

LOGARITHM TABLES

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N.	0	1	2	3	4	5	6	7	8	9
940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678
950	977724	977769	977816	977861	977906	977952	977998	978043	978089	978135
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	986772	986817	986861	986906	986951	986996	987040	987086	987130	987175
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

N.	0	1	2	3	4	5	6	7	8	9
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Section IX
ANSWERS TO PROBLEMS

ALGEBRA

(Answers to Problems—Pages 215 to 234 Inclusive)

Positive and Negative Numbers

1. (a) 2; (b) 4; (c) 2; (d) -4; (e) 18; (f) -4; (g) .22; (h) 48.
2. (a) 2; (b) -2; (c) 8; (d) -8; (e) -5; (f) -5.
3. (a) -24; (b) 1680; (c) -72; (d) -560.
4. (a) -4; (b) 4; (c) -9; (d) 7; (e) -35; (f) -24.

Common Fractions

5. (a) 1; (b) 1 5/12; (c) 2 13/60; (d) $\frac{3x+2y+z}{12}$
6. (a) 1/5; (b) 1/2; (c) 1/8; (d) $\frac{3x-5y}{15}$
7. (a) 1/3; (b) 5/2; (c) 1; (d) 1/2.
8. (a) 3/4; (b) 1/8; (c) 3/5; (d) 2.
9. (a) 2/3; (b) 1/3; (c) 5/6; (d) a/b .
10. (a) $\frac{227}{360}$; (b) 1/4; (c) $\frac{bx+ay}{ab}$; (d) $\frac{bx-ay}{ab}$
11. (a) 17/4; (b) 11/8; (c) 13/3; (d) 8/3.
12. 5/32.
13. 3 3/32.
14. $2 \frac{53}{56}$ sq. in.
15. 2830 pounds.

Decimal Fractions

16. (a) .75; (b) .50125; (c) 5.56; (d) 37.
17. (a) 1.375; (b) 2.6875; (c) .899; (d) 1.35; (e) .125; (f) 2.375.
18. (a) .28125; (b) .7854; (c) 4.0625; (d) 52.095.
19. (a) .7854; (b) 40; (c) 7.25; (d) 22.5.
20. (a) .333; (b) .25; (c) .20; (d) 1667; (e) .14286; (f) 125; (g) 111; (h) .0625; (i) .03125; (j) .01563.
21. 3.927 in. 2 $\frac{3}{4}$ 680 pounds. 25 2800 pounds
22. 2.91525 sq. in. 24. 281 pounds.

Square Root

26. (a) 132; (b) 3361; (c) 2; (d) 4; (e) $\sqrt[4]{4}$.
27. 3 in. 29. 29.15 in. 31. 22.45 in.
28. 35.2 in. 30. 10 in.

Exponents

32. (a) x^2 , 9; (b) x^3 , 8; (c) x^5 , 32; (d) x , 3.
 33. (a) $3.1 x^2$; (b) $4x^2$; (c) $125y^3$; (d) $-12y$.
 34. (a) x^7 ; (b) x^{14} ; (c) x^3 ; (d) $120x^{10}$; (e) $\frac{3}{4}x^{10}$.
 35. (a) x^3 ; (b) x^3 ; (c) $4x^2y$; (d) $2x^3y$.
 36. (a) a ; (b) $6y$; (c) $4a^2b^2$.
 37. (a) $x^2 + 2xy + y^2$; (b) $1.414x^2$; (c) a^5 ; (d) x^7 ; (e) 72; (f) -55.
 38. (a) x ; (b) 2; (c) $1/x^2$; (d) a^2 ; (e) $x^{3.5}$.
 39. (a) 64; (b) $7^{4/3}$; (c) 4; (d) $a^{8/5}$.
 40. (a) x^3y^4 ; (b) $81a^3$; (c) $\frac{1}{a^2b^2}$; (d) 1; (e) $\frac{a^2}{b^2}$; (f) $\frac{y^2}{x^3}$; (g) a^5
 41. (a) w^3 ; (b) 216. 42. .1841.

Solution of Equations

- | | | |
|--------|-------|-----------|
| 43. 2 | 50. 8 | 57. 11/2 |
| 44. 15 | 51. 1 | 58. 5 |
| 45. 8 | 52. 5 | 59. 5 |
| 46. 7 | 53. 5 | 60. 5 |
| 47. 10 | 54. 1 | 61. 0 |
| 48. -2 | 55. 4 | 62. 11 |
| 49. 8 | 56. 9 | 63. 54.77 |

Simplification

- | | | |
|---|--|---|
| 64. $x = 40$. | 66. $x = -1$. | |
| 65. $x = 1 \frac{13}{17}$ | 67. (a) $x = 11$.
(b) $x = 12 + 11b$. | |
| 68. (a) $x = -\frac{6}{35}$; (b) $x = -3$; (c) $x = -3/2$ | | |
| 69. (a) $x = 1 - 3/2$; (b) $x = -4(1 + a)$. | | |
| 70. (a) $x = 48/17$; (b) $x = -10 \pm 1/10\sqrt{161}$ | | |
| 71. $fb = \frac{6M}{bb^2}$ | 75. .140 sq. in. | 81. $x = 2, 1$ |
| 72. $AR = \frac{b}{c}$ | 76. 425 pounds. | 28. $x = 5$ |
| 73. 30° | 77. 252 pounds. | 83. $x = 1$ |
| 74. 104° | 78. $4\frac{1}{3}$ | 84. $x = 3$ |
| 86. $x = -2, x = -2$ | 79. $25/9$ | 85. $\frac{-1 \pm \sqrt{11}}{2}$ |
| 87. $x = \pm 2.499; x = 0$ | 80. $x = \pm 1$ or ± 2 | |
| 88. $x = \pm 3.162; x = 0$ | | 91. $x = 2.305; x = .715; x = -3.025$ |
| 89. $x = 1.405; x = .58$ | | 92. $x = 1.59; x = 4.42; x = -2$ |
| 90. $x = 1.355$ | | 93. $x = -1; x = 4$ |
| | | 94. $x = 3; x = -1; x = 1 \pm \sqrt{6}$ |

Factoring

- | | | |
|---------------------------------|----------------------------|------------------------|
| 95. -1 | 99. $\frac{2}{3}, 1$ | 103. $2x^2 - x - 3$ |
| 96. -1, 4. | 100. $-4/5, -3$ | 104. $4n^2 + 4n - 3$ |
| 97. $-\frac{1}{2} \pm \sqrt{3}$ | 101. $4/5, 3$ | 105. $5y^2 + 28y + 15$ |
| 98. $-\frac{1}{2}$ | 102. $1, -\frac{1}{2}, -3$ | |

106. $8x^3 + 38x^2 + 12x - 90$
 107. $9x^2 + 24x + 16$
 108. $x^2 + 15x^2 + 75x + 125$
 109. $3x + 9$
 110. $x^2 - 2x\sqrt{2x+3} + 2x + 3$
 111. $2x^2 + 2x\sqrt{x^2 + 4x + 3} + 4x + 3$
 112. $3x^2 - 7x + 2$
 113. $5x^3 + 2x^2 - 6x + 3$
114. $\frac{7x^2 + 3x + 9}{x - 2}$
 115. $3x + 4$
 116. $2x^4 - \frac{5x^3}{3} + \frac{2x^2}{9} + \frac{3x}{12} - \frac{1}{8}$
 117. $\frac{b^4}{18} - \frac{5b^3}{36} - \frac{b^2}{60} + \frac{2b}{15} + \frac{1}{25}$
 118. 3, 6
 119. 5.5 sq. in.

Completing the Square

120. $x = 3$ or -13
 121. $x = 3$ or 2
 122. $x = \frac{9}{2}$ or $-\frac{11}{2}$
 123. $x = \frac{1}{6}$ or -1
 124. $x = 2$ or $\frac{3}{2}$
 125. $x = 11$ or $\frac{11}{3}$
 126. $x = \frac{1}{2} \pm \sqrt{\frac{1}{2}}$
 127. $x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{\frac{4^3}{3}}$
 128. $x = -\frac{1}{3} \pm \sqrt{\frac{2}{3}}$
 129. $\frac{3}{2}$ or $-\frac{4}{3}$
 130. 18.96
 131. 16.225
 132. 17.32
 133. 13.86
 134. $\frac{L}{\rho} = 65$
 135. .226
 136. $x = -2 \pm \sqrt{3}$
 137. $x = 1$ or -3
 138. $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$
 139. $x = \frac{3}{4}$ or $-\frac{1}{4}$
 140. $x = \frac{1}{2} \pm \frac{3}{2}\sqrt{\frac{3}{5}}$
 141. $x = 4 \pm \sqrt{97}$
 142. $x = 0.15 \pm 1.331$
143. $x = 4$ or $-\frac{17}{4}$
 144. $x = 1.80$ or -0.55
 145. $x = 348$ or -0.315
 146. $x = 9$
 147. $x = 3$
 148. $x = 10$, $x = -2$
 149. $x = 10$
 150. $x = -1$
 151. $x = \frac{-15 \pm \sqrt{45}}{18}$
 152. $x = 6$
 153. $x = 22$, $x = 10$
 154. $x = -2$
 155. $x = -9$
 156. $x = 3$, $y = 1$
 157. $x = 2\frac{1}{2}$, $y = \frac{1}{4}$
 158. $x = 2$, $y = 4$
 159. $x = 6$, $y = 2$
 160. $x = -15$, $y = 15$
 161. $x = -2$, $y = 5$
 162. $x = 2$, $y = 1$
 163. $x = \frac{4}{5}$, $y = -\frac{3}{5}$
 164. $x = 2$, $y = \frac{1}{2}$
 165. $x = \pm 3$, $y = 4$
 166. $x = 2$, $y = -3$
 167. $y = 1$, $x = 2$
 168. $x = 3,000$, $y = 1,000$
 169. $A = -100$, $B = 173.21$
 170. $x = \pm 4$, $y = \pm 6$
 171. $x = 7$, $y = -5$
 $x = -5$, $y = 7$
 172. $x = \pm 4$, $y = \pm 3$
 173. $x = \frac{1}{4}$, $y = 1$

174. $x = \frac{1}{2}, y = -\frac{1}{3}, z = \frac{1}{4}$

192. $x = \frac{99}{24}, y = \pm \frac{9}{8} \sqrt{15}$

175. $x = \frac{5}{2}, y = \frac{1}{2}, z = \frac{3}{2}$

193. $x = 1 \pm 2\sqrt{2}, y = 4 \pm 2\sqrt{2}$

176. $x = 2, y = 3$

194. $x = 10, y = \frac{1}{9}$

177. $R_1 = 12\frac{1}{2}, R_2 = 37\frac{1}{2}$

195. $A = 16, B = -10$

178. $x = \pm 2, y = \pm 1$

196. $x = \frac{9}{5}, y = \frac{4}{5}, z = \frac{17}{5}$

179. $x = 2, y = 3$

197. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{4}$

180. $x = 2, y = 3$

198. $x = \frac{37}{16}, y = \frac{55}{16}, z = -\frac{27}{16}$

181. $x = \frac{2(B-A)}{2B-A}$

$$x = \frac{AB}{2B-A}$$

182. $T = 86.6, c = -50$

199. $x = \frac{1}{2}, y = -\frac{1}{3}, z = \frac{1}{4}$

183. $x = 11, y = 12, z = 13$

200. $x = 3, y = 6, z = 1$

184. $x = -\frac{19}{7}, y = -\frac{10}{17}, z = -1$

201. $x = 2, y = 8$

185. $x = 1, y = -1, z = 2$

202. $A = 5, B = 4$

186. $x = 2, y = -4$

203. $S = 484, T = 11$

187. $x = -1 \pm 2\sqrt{2}, y = 8 \pm 4\sqrt{2}$

204. $x = 4, y = 7$
 $x = -1, y = 2$

188. $S = 484, T = 11$

205. $\tan \theta = \frac{V^2}{GR}$

189. $x = \pm \frac{80}{\sqrt{247}}, y = \pm \frac{15}{\sqrt{247}}$

206. $\alpha = 45^\circ, \beta = 75^\circ$

190. $x = 4 \text{ or } -\frac{12}{23}$

207. $\alpha = 51^\circ, \beta = 39^\circ$

$$y = -14 \text{ or } \frac{198}{23}$$

208. $A = 98\frac{1}{3}, B = 36\frac{2}{3}, C = 45^\circ$

191. $x = 32.6, y = 38.8$

209. $Dx = 2, Dy = 4, Dz = 6$

210. $Dx = 8.5513, Dy = 4.8651, Dz = 7.9411$

Copper	7.266
Manganese	2.595
Magnesium	0.865
Aluminum	162.274
	173.000

Copper	6.96
Manganese	0.87
Magnesium	0.87
Aluminum	165.30
	174.00

213. 2.8 inches

215. 31.25%

217. 20.60%

214. 65.52%

216. 6.25%

218. Material

Density

S. G.

Water 62.4

1.00

Spruce 27

.43

Mg 109

1.74

24ST' 173

2.77

17ST' 174

2.78

Steel 490

7.85

219. 2.83

220. 27.50

221. 11.67

222. .25

223. (a) 76.2cm;
(c) 22.83 in.;

(b) 64.516cm
(d) 7.87 in.

224.	(a) .196 in. ² , (c) 19.6 in. ² ;	(b) 4.5 in. (d) 7.1 in.
225.	(a) 146.67 ft./sec.; (c) 58.66 ft./sec.;	(b) 158 m.p.h. (d) 122.73 m.p.h.
226.	(a) 29.92 in.; (c) 203.6 in.,	(b) 20.63 lbs./in. ² (d) 12.28 lbs./in. ²
227.	(a) .5235 Radians (c) 42.97°	(b) 1.0 Radians (d) 68.76°
228.	(a) 16.09 kilo (c) 926.99 kilo	(b) 15.53 miles (d) 1895.27 miles
229.	450 r.p.m.	.140625
230.	1250 r.p.m.	3231 lbs.
231.	33.3 cu. feet	238. (a) 9.32; (b) 31.06; (c) 3106
232.	180	239. (a) 44.7; (b) 400, (c) 346
233.	67.4	240. $L_{100} = 1189$
234.	331.1	$L_{150} = 2675$
235.	880 feet	$L_{200} = 4756$

GEOMETRY

(Answers to Problems — Page 235)

1. $37^{\circ} 18' 5''$	9. $12^{\circ} 44' 49\frac{1}{3}''$	15. (a) .244
2. $161^{\circ} 17' 1''$		(b) 1.57
3. $9^{\circ} 19' 57''$		(c) 2.62
4. $39^{\circ} 50' 59''$	10. $42^{\circ} 52' 15\frac{7}{8}''$	(d) 104.6°
5. $8^{\circ} 26' 16'$		(e) 90°
6. $167^{\circ} 24'$	11. 1 Radian	(f) 720°
7. $146^{\circ} 55' 9''$	12. 62,832	16. $57^{\circ} 17' 45''$
8. $7^{\circ} 17' 13''$	13. 4	17. $28\frac{6}{7}$
		18. 90

TRIGONOMETRY

(Answers to Problems — Pages 235 to 240 Inclusive)

Types of Triangles

- Right and oblique triangles
- A triangle with one angle equal to 90° .

Elements of Triangles

- Three sides and three angles.
- (a) Protractor
(b) Construction with compass and triangle
(c) Equation $R = 180^{\circ} - (A + B)$; If the remainder is 90° the triangle is a right triangle.

Trigonometric Functions in Right Triangles

5. $\sin \phi = \frac{o}{b}$

$\cos \phi = \frac{a}{b}$

$\tan \phi = \frac{o}{a}$

6.

FUNCTION	RECIPROCAL FUNCTION
SIN	COSECANT
COS	SECANT
TAN	COTANGENT

7. Cosecant $\phi = \frac{b}{o}$

Secant $\phi = \frac{b}{a}$

Cotangent $\phi = \frac{a}{o}$

Geometric Relations

8. $b^2 = a^2 + o^2$ Refer to Problem 5)

9. (a) $a = 15$; $b = 20$; $c = 25$

(b) $a = 15$; $b = 6$; $c = 16.1^+$

(c) $a = 5$; $b = 6$; $c = 7.8^+$

(d) $a = 5$; $b = 8.6^+$; $c = 10$

10. Hypotenuse = 20

Use of Table of Natural Trigonometric Functions—Interpolation

11. 0.17655

17. 30° (a) Sin .5000

12. 0.86155

(b) Cos .8660

15. (a) $36^\circ 52' 12''$

(c) Tan .5774

(b) $53^\circ 7' 48''$

18. 45° (a) Sin .7071

16. (a) $\sin \phi = .6000$

(b) Cos .7071

$\cos \phi = .8000$

(c) Tan 1.0000

$\tan \phi = .7500$

19. (a) 95.2655

(b) $\sin \phi = .8000$

(b) 129.9075

$\cos \phi = .6000$

(c) 164.5495

$\tan \phi = 1.3333$

13. 1.2282

20. 6 ft. 9.067 in.

14. $10^\circ 45' 30''$

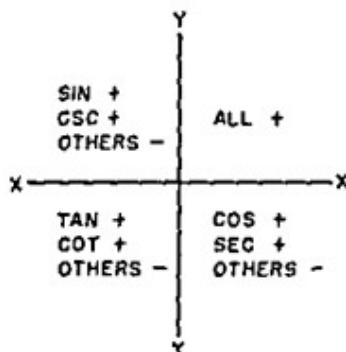
Trigonometric Function in Oblique Triangles

21. (Solve graphically)

22. 15° 1st Quadrant
 108° 2nd Quadrant
 210° 3rd Quadrant
 500° 4th Quadrant

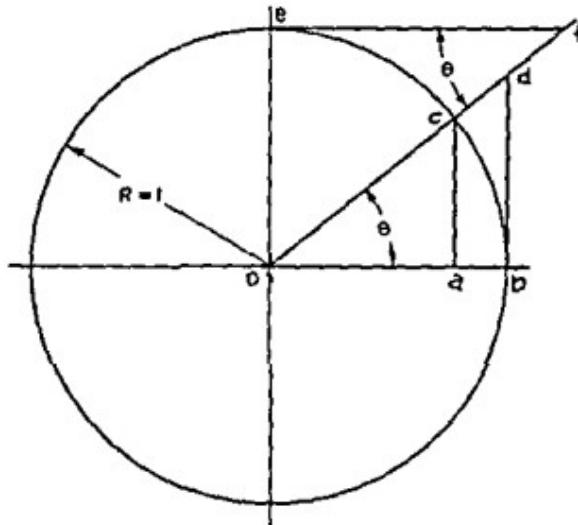
23.

	SIN	COS	TAN	COTAN	SEC	COSEC
15°	+.2588	.9659	+.2679	+3.7321	+1.0353	+3.8637
108°	+.9511	-.3090	-3.0777	-.3249	-3.2361	+1.0515
210°	-.5000	-.8660	+.5774	+1.7321	-.11547	-2.0000
300°	-.8660	+.5000	-1.7321	-.5774	+2.0000	-1.1547



Geometric Representation of the Trigonometric Functions

24



$$\sin = \frac{cd}{co} = \frac{ca}{1} = ca$$

$$\cos = \frac{ca}{oc} = \frac{ca}{1} = ca$$

$$\tan = \frac{bd}{ob} = \frac{bd}{1} = bd$$

$$\csc = \frac{ef}{oe} = \frac{ef}{1} = ef$$

$$\sec = \frac{od}{ob} = \frac{od}{1} = od$$

$$\csc = \frac{of}{oe} = \frac{of}{1} = of$$

Value of the Functions of Obtuse Angles

- $$\begin{array}{lll} \text{25. } \sin 100^\circ = +.98481 & \sin 200^\circ = -.34202 & \sin 300^\circ = -.86603 \\ \cos 100^\circ = -.17365 & \cos 200^\circ = -.93969 & \cos 300^\circ = +.50000 \\ \tan 100^\circ = -5.6713 & \tan 200^\circ = +.36397 & \tan 300^\circ = -1.7321 \end{array}$$

Oblique Triangles Solved As Right Triangles

26.

	A	B	C	d	b	c
a)	28°	101°	51°	15.3	32	25.3
b)	21°	153°	6°	57	72.2	16.6
c)	64°	67°	49°	15.5	15.9	13
d)	31° 27'	105° 30'	43° 3'	39	72	51
e)	0°	0°	180°	10	25	35
f)	16°	149° 7'	14° 53'	16.5	36	18
g)	32° 9'	128°	19° 51'	99	146.6	63

Oblique Triangles Solved By Special Formulas

27. Same answers as in Problem 26.

Trigonometric Formulas

30. (a) $\frac{\sin \phi + 1}{\cos \phi}$

$$(b) \quad \frac{24}{25}; \frac{7}{25}; \frac{24}{7}$$

$$(b) \quad \sin^2 \phi + \frac{1}{\cos^2 \phi}$$

$$(c) \quad \frac{120}{169}; -\frac{119}{169}; \frac{120}{119}$$

$$(c) \quad \frac{\cos^2 \phi}{\sin^2 \phi} + \frac{1}{\sin^2 \phi}$$

$$40. \quad (a) \quad -\frac{3}{\sqrt{13}}; -\frac{2}{\sqrt{13}}; -\frac{3}{2}$$

$$(d) \quad \sin \phi + \frac{1}{\cos \phi}$$

$$(b) \quad -\frac{2}{\sqrt{13}} ; \quad \frac{3}{\sqrt{13}} ; \quad -\frac{2}{3}$$

$$(e) \sin \phi \cos \phi$$

$$42 \quad (a) \quad \sin 7\phi = \sin^3 \phi$$

$$31. \quad (a) \quad \frac{1 + \sin^2 \phi}{1 - \sin^2 \phi}$$

42. (a) $\sin \gamma \phi - \sin \beta \phi$
 (b) $\frac{1}{2} \cos 3\phi + \frac{1}{2} \cos \phi$
 (c) $\frac{1}{2} \sin 11\phi - \frac{1}{2} \sin \phi$

$$(b) \frac{1}{\sin \phi (1 - \sin^2 \phi)}$$

$$43. \quad (a) \quad -2 \cos 45^\circ \sin 1^\circ$$

55. (a) -0.2797
 (b) 0.0098

44. (a) $\cot 30^\circ \tan 10^\circ$
 (b) $2 \sin 2\phi \sin \phi$

$$36. \quad (a) \quad \frac{56}{65}$$

$$(b) -\tan \frac{3\phi}{2} \tan \frac{\phi}{2}$$

$$(b) - \frac{33}{65}$$

46. $A = 43^\circ$
 $B = 50^\circ$ (to the nearest degree)

$$(c) -\frac{36}{77}$$

$$C = 87^\circ$$

$$39. \text{ (a)} \quad \frac{24}{25}; \frac{7}{25}; \frac{24}{7}$$

- (b) 172,695.9
- (c) 1,429,431.7

LOGARITHMS

(Answers to Problems—Pages 240 to 244 Inclusive)

Introduction

1. (a) Multiplication (c) Raising to a power
(b) Division (d) Extracting a root

Definitions and Principles

- | Definitions and Principles | | | |
|----------------------------|-------------------------------|---------------|-------------------------|
| 2. | (a) 3^{10} | (e) 3^{42} | |
| | (b) 10^1 | (f) 3^0 | |
| | (c) 5^2 | (g) 3^2 | |
| | (d) 10 | (h) $8^{7/3}$ | |
| 5 | (a) 3 = <i>characteristic</i> | (b) | 00000 = <i>mantissa</i> |
| 6. | (a) Mantissa | (b) | Positive |
| 7 | (a) $\overline{4}30103$ | (f) | 1 30103 |
| | (b) $\overline{3}30103$ | (g) | 2 30103 |
| | (c) $\overline{2}30103$ | (h) | 3 30103 |
| | (d) $\overline{1}30103$ | (j) | 4 30103 |
| | (e) 0 30103 | | |
| 8 | (a) $\overline{3}66032$ | (c) | -3.17173 |
| | (b) $\overline{10}15320$ | (d) | -0.9934 |

Rules for Characteristics

Use of Tables of Logarithms

- | | | | |
|-----|--------------------------|-------------------------|-------------------------|
| 14. | (a) 2.54531 | | (f) $\overline{1}34635$ |
| | (b) 1.77525 | | (g) $\overline{3}80618$ |
| | (c) 0.99957 | | (h) 3 00945 |
| | (d) 0.15836 | | (i) 6 30103 |
| | (e) $\overline{1}87448$ | | |
| 15. | (a) 1005. | (j) 2 30081 | |
| | (b) 742 0 | (f) 0 016964 | |
| | (c) 7.95 | (g) 0 000,000,297,18 | |
| | (d) 43,700,000 | (h) 0 003,894 | |
| | (e) 0 2400 | (i) 89.73 | |
| 16. | (a) $\overline{2}47666$ | (j) 998 1 | |
| | (b) 0.32504 | (f) 0 890294 | |
| | (c) 1.518618 | (g) $\overline{3}82347$ | |
| | (d) $\overline{1}602798$ | (h) 6 360704 | |
| | (e) 0 076388 | (i) 5 9666624 | |
| | | (j) 0 426302 | |

17.	(a) 187.066 (b) 12.3257 (c) 0.052885 (d) 0.0006763666 (e) 0.5102555	(f) 45,445,550. (g) 94,417.5 (h) 1.035098 (i) 2.005727 (j) 0.2980214
-----	---	--

Fundamental Operations Using Logarithms

18.	(a) 7,600 (b) 9,240 (c) 7,736 (d) 86,718 (e) 0.6776	(f) 22.9824 (g) 2,280,912 (h) 0.000,054,08 (i) 4.2464 (j) 23,478
19.	(a) 1.0575 (b) .5238 (c) 1.2766 (d) 17.0726 (e) 14.5	(f) 0.04762 (g) 0.00330 (h) 1.0767 (i) 0.6932 (j) 15.0
20.	(a) 390,625 (b) 46,656 (c) 86,436 (d) 18.3788 (e) 907,039,232	(f) 3.68715 (g) 58.064,400 (h) 0.001331 (i) 0.000,000,008 (j) 792.32
21.	(a) 9.1652 (b) 83.318 (c) 10.03326 (d) 7.55625 (e) 0.344383	(f) 16.4924 (g) 4.44796 (h) 3.680 (i) 2.64575 (j) 31.62286

Cologarithms

22. A cologarithm is the logarithm of the reciprocal of a number.

Division or Multiplication of Logarithms

24. No			
25.	(a) 0.74124 (b) 2.45065	(c) 3.02985	(d) .97346

Solution of Equations Using Logarithms

26.	(a) .24444	(b) .0322023	(c) 454.2333
27.	(a) .03052 (b) .023298	(c) .000,007,23	(d) .0007645
28.	(a) 1,758,520	(b) .779466	(c) .008802
29.	(a) 3502.78	(b) 22.1843	(c) 69.7091
30.	(a) $a = 61.8$; $b = 102.86$; $B = 59^\circ$ (b) $b = 7948$; $c = 7971.6$; $B = 85^\circ 25'$ (c) $b = 2.221$; $c = 3.118$; $A = 44^\circ 35'$ (d) $a = 13.69$; $c = 21.77$; $A = 38^\circ 58'$ (e) $b = 1.468$; $A = 26^\circ 5'$; $B = 63^\circ 55'$		
31.	(a) $b = 53.48$; $c = 54.30$; $C = 67^\circ 23'$ (b) $a = 1222$; $c = 1297$; $C = 75^\circ 33'$ (c) $a = 9.368$; $b = 0.1810$; $C = 110^\circ 17'$ (d) $b = 4017$; $c = 2217$; $B = 85^\circ 11'$		

ANALYTICAL GEOMETRY OF STRAIGHT LINES

(Answers to Problems — Pages 244 to 247 Inclusive)

2. $y = 6$ Straight line parallel to x axis
 $y = 22$ Straight line parallel to x axis.
 $x = 10$ Straight line parallel to y axis

3. Origin

5. A linear equation as all its plotted points fall on a straight line

6. Yes

10. Positive

11. Negative

12. (a) $x = -2, y = 4$
(b) $x = 10, y = -20$
(c) $x = 8, y = 2 \frac{2}{3}$
(d) $x = -1, y = 1$

13. (a) Slope + 2
(b) Slope + 2
(c) Slope $-\frac{4}{3}$
(d) Slope + 1

14. (a) $\frac{r}{400-x} = \frac{30}{400}$
(b) 21.75

15. Parallel lines

16. At right angles

17. Product

18. (a) Parallel
(b) Parallel
(c) Perpendicular
(d) Neither

19. 58.6 minutes

20. $73^{\circ} 34'$

21. (a) $y = 2x + 4$
(b) $y = 7x - 28$
(c) $y = -66x$
(d) $y = x - 22$
(e) $y = -2x - 11$

22. (a) $y = 9x + 2$
(b) $y = -2x + 21$
(c) $y = 2x + 6$
(d) $y = -4x + 49$
(e) $y = x - 2$

23. No

24. Yes

- | | | |
|----------------------|-----------------------------|-----------------|
| 26. (a) Inconsistent | | (c) Independent |
| (b) Inconsistent | | (d) Independent |
| 28. (a) 10.08 | (b) 3.204 | (c) 7.25 |
| 29. $23\frac{1}{2}$ | 30. (a) -3
(b) -1.45 | |

APPENDIX — SLIDE RULE

(Answers to Problems — Pages 247 to 249 Inclusive)

Division

1. .250	8. .300	15. 87.38
2. 4	9. .899	16. 16.47
3. .549	10. 2.236	17. .0000420
4. 1.822	11. 2.777	18. 25
5. .884	12. 1.436	19. .04
6. 5.333	13. .314	20. .00215
7. .188	14. 3.185	

Multiplication

21. 3.40	26. .75	31. 774
22. 4.03	27. .075	32. .1073
23. 750	28. 7.5	33. .00617
24. 7500	29. .00075	34. 2687
25. 7500	30. 77.4	

35.	16	17	18	19	20	21	22	23	24	
	40	640	680	720	760	800	840	880	920	960
	41	656	697	738	.779	820	861	902	943	984
	42	672	714	756	798	840	882	924	966	1008
	43	688	731	774	817	860	903	946	989	1032

Combined Multiplication and Division

36. 1.80	40. 119.8	44. 1.019
37. .51.76	41. .0473	45. 961.6
38. 10.11	42. 1.104	
39. 3101	43. 3.246	

Proportions

46. 50.67	50. 46.62	54. 1.100
47. 105.4	51. 2.174	55. 74.14
48. 91.59	52. 5.266	56. .00005016
49. 13,750	53. 1.464	

Square Roots and Squares of Numbers

Find the square root of the following:

57.	.01477	61.	6611	65.	913.2
58.	.4669	62.	.1367	66.	81.36
59.	4.669	63.	.05385	67.	19.39
60.	2961	64.	3048		

Find the squares of the following.

68.	9.672	73.	15.92	78.	.0000005213
69.	80.28	74.	1369	79.	1,040,400
70.	10,000	75.	213,400	80.	2,490.0
71.	3,856	76.	00001689	81.	597,500
72.	3168	77.	.08123	82.	29.48

Square Root of the Sum or the Difference of Two Squares

83.	123.7	85.	101.4	87.	99.50
84.	92.66	86.	76.92		

Cube Roots and Cubes of Numbers

88.	1260	95.	.02840	102.	125,000
89.	2277	96.	4064	103.	599,100
90.	4135	97.	4642	104.	2414
91.	1262	98.	1,368	105.	1,000,000
92.	70	99.	29,790	106.	7881
93.	2789	100.	103,800	107.	6745
94.	.2611	101.	110,600		

Trigonometric Functions

108.	1.0	114.	986286	120.	.974370
109.	.707107	115.	.002909	121.	.999837
110.	.011926	116.	948324	122.	.584250
111.	.182808	117.	342020	123.	.936672
112.	.019197	118.	173648	124.	.039260
113.	.148097	119.	.258819		

Logarithms

125.	0.1584	129.	26164	133.	2.960
126.	1.7275	130.	118.6	134.	5.972
127.	9.5011 — 10	131.	9,510,000,000	135.	23.15
128.	8.8585 — 10	132.	15,370,000,000		

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